"The Price of Doing Business: How Upfront Costs Deter Political Risk"

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Online Appendix

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Proposition 1. For any given takings rate in an industry j, $\tau_j \geq 0$, there exist types x_{ji} and y_{ji}, for each industry j and $i \in \{d, f\}$, such that $0 < x_{ji} < y_{ji}$. Firms that are in the market decide to exit if $\varphi < x_{ji}$, and stay and produce if $x_{ji} \leq \varphi$. Firms that are out of the market decide to stay out if $\varphi < y_{ji}$, and enter and produce if $y_{ji} \leq \varphi$.

Proof of Proposition 1.

1.1 Consumption

Recall that:

$$U = \left(1 - \sum_{j=1}^{J} w_j\right) \log v_0 + \sum_{j=1}^{J} w_j \log Q_j \qquad \text{where: } Q_j \equiv \left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}}$$

If the price of a variety in industry j is $p_j(v)$ and the quantity consumed of a variety in industry j is $q_j(v)$, then the budget constraint is:

$$\sum_{j=0}^{J} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv \leq R$$

where R is the total revenue.

The Lagrangian is:

$$\mathcal{L} = \left(1 - \sum_{j=1}^{J} w_j\right) \log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda \left[R - \sum_{j=0}^{J} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv\right]$$

Then constrained optimization for industry $j \in \{1, \ldots, J\}$ and product v' yields:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_j(v')} &= \frac{w_j}{Q_j}\left(\frac{\sigma}{\sigma-1}\right)\left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}-1}\left(\frac{\sigma-1}{\sigma}\right) q_j(v')^{\frac{\sigma-1}{\sigma}-1} - \lambda p_j(v') \\
&= w_j \left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{-1} q_j(v')^{-\frac{1}{\sigma}} - \lambda p_j(v')
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= R - \sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv \\
\lambda &\geq 0 \\
\lambda \frac{\partial \mathcal{L}}{\partial \lambda} &= \lambda \left[R - \sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv \right] = 0
\end{aligned}$$

Note that if $\lambda = 0$, then $\frac{\partial \mathcal{L}}{\partial q_j(v)} > 0$ for all levels of consumption. So it must be true that $\lambda > 0$ and $R - \sum_{j=0}^{J} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv = 0$.

Additionally, note that for any industry j and goods v, $v' \in V_j$, we have:

$$\begin{aligned}
\lambda &= w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v)^{-\frac{1}{\sigma}} p_j(v)^{-1} = w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} \\
&\Leftrightarrow \quad q_j(v)^{-\frac{1}{\sigma}} = q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} p_j(v) \\
&\Leftrightarrow \quad q_j(v)^{\frac{\sigma-1}{\sigma}} = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} p_j(v)^{1-\sigma} \\
&\Leftrightarrow \quad \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} \int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \\
&\Leftrightarrow \quad q_j(v')^{\frac{\sigma-1}{\sigma}} = p_j(v')^{1-\sigma} \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{-1} \\
&\Leftrightarrow \quad q_j(v') = p_j(v')^{-\sigma} \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{-\frac{\sigma}{\sigma-1}} \\
&\Leftrightarrow \quad q_j(v') = p_j(v')^{-\sigma} Q_j P_j^{\sigma} \quad \text{where: } P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}
\end{aligned}$$

So the demand function for any variety v in any industry j is:

$$q_j(v) = p_j(v)^{-\sigma} Q_j P_j^{\sigma}$$

where:

$$Q_j \equiv \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}$$

1.2 Production

Let the wage be 1, which represents one unit of the numeraire good, which is the unique good produced in industry $j = 0$. We suppress industry notation.

To make an output of q units, a domestic firm must use labor, $\frac{q}{\varphi}$, where φ is the firm's productivity. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by domestic firms is:

$$\pi_d(\varphi) = p_d(\varphi)\, q_d(\varphi) - \left[\frac{q_d(\varphi)}{\varphi} + c\right] = p_d(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_d(\varphi)^{-\sigma} QP^{\sigma}}{\varphi} + c\right]$$

Profit maximization by domestic firms yields:

$$\frac{\partial \pi_d(\varphi)}{\partial p_d} = (1-\sigma)\, p_d(\varphi)^{-\sigma} QP^{\sigma} + \sigma \left[\frac{p_d(\varphi)^{-\sigma-1} QP^{\sigma}}{\varphi}\right] = 0$$

$$\Leftrightarrow p_d(\varphi) = \frac{\rho}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma - 1}$$

This yields revenue:

$$r_d(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_d(\varphi) &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma} - \left[\frac{\left(\frac{\varphi}{\rho}\right)^{\sigma} QP^{\sigma}}{\varphi} + c\right] \\ &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}\left(1 - \frac{1}{\rho}\right) - c = \frac{r_d(\varphi)}{\sigma} - c\end{aligned}$$

To make an output of q units, a foreign firm must use labor, $\frac{q(1+\tau)}{\varphi}$, where φ is the firm's productivity and τ is the government taking rate. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by foreign firms is:

$$\pi_f(\varphi) = p_f(\varphi)\, q_f(\varphi) - \left[\frac{q_f(\varphi)(1+\tau)}{\varphi} + c\right] = p_f(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_f(\varphi)^{-\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right]$$

Profit maximization by foreign firms yields:

$$\frac{\partial \pi_f(\varphi)}{\partial p_f} = (1-\sigma)\, p_f(\varphi)^{-\sigma} QP^{\sigma} + \sigma \left[\frac{p_f(\varphi)^{-\sigma-1} QP^{\sigma}(1+\tau)}{\varphi}\right] = 0$$

$$\Leftrightarrow p_f(\varphi) = \frac{\rho(1+\tau)}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma - 1}$$

This yields revenue:

$$r_f(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_f(\varphi) &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma} - \left[\frac{\left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right] \\ &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}\left(1 - \frac{1}{\rho}\right) - c = \frac{r_f(\varphi)}{\sigma} - c\end{aligned}$$

So conditional on being in the market, optimal production by domestic firms yields:

$$p_d^*(\varphi) = \frac{\rho}{\varphi} \quad \text{and} \quad q_d^*(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma} QP^{\sigma} \quad \text{and} \quad r_d^*(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}$$

This yields domestic firm profit:

$$\pi_d^*(\varphi) = \phi_d \varphi^{\sigma-1} - c = \frac{r_d^*(\varphi)}{\sigma} - c \qquad \text{where: } \phi_d \equiv \frac{QP^{\sigma}}{\sigma \rho^{\sigma-1}}$$

And optimal production by foreign firms yields

$$p_f^*(\varphi) = \frac{\rho(1+\tau)}{\varphi} \quad \text{and} \quad q_f^*(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} QP^{\sigma} \quad \text{and} \quad r_f^*(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}$$

This yields foreign firm profit:

$$\pi_f^*(\varphi) = \phi_f \varphi^{\sigma-1} - c = \frac{r_f^*(\varphi)}{\sigma} - c \qquad \text{where: } \phi_f \equiv \frac{QP^{\sigma}}{\sigma \rho^{\sigma-1}(1+\tau)^{\sigma-1}}$$

1.3 Firm Entry and Exit

We suppress industry notation. First, note the following properties of the profit function:

$$\lim_{\varphi \to 0} \pi_i(\varphi) = -c < 0 \quad \text{and} \quad \lim_{\varphi \to \infty} \pi_i(\varphi) = \infty > 0$$

$$\text{and} \quad \frac{\partial \pi_i(\varphi)}{\partial \varphi} = (\sigma - 1)\phi_i \varphi^{\sigma-2} > 0$$

So function $\pi_i^*(\varphi)$ is invertible: for any $y \in \mathbf{R}$, there exists a unique $\varphi_y \in (0, \infty)$ such that $\pi_i^*(\varphi_y) = y$.
Define: $\Psi_i \equiv V_i^{in} - V_i^{out}$.
A firm that is already in the market has incentive to stay in the market (rather than exit) iff:

$$\mu_i \kappa_i + \delta V_i^{out} \leq \pi_i^*(\varphi) + \delta V_i^{in} \quad \Leftrightarrow \quad \mu_i \kappa_i - \delta \Psi_i \leq \pi_i^*(\varphi)$$

Define: $x_i \equiv \pi_i^{*-1}(\mu_i \kappa_i - \delta \Psi_i)$. This is the type of firm that is in the market and indifferent about whether to remain.
A firm that is not already in the market has incentive to enter the market iff:

$$\delta V_i^{out} \leq \pi_i^*(\varphi) - \kappa_i + \delta V_i^{in} \quad \Leftrightarrow \quad \kappa_i - \delta \Psi_i \leq \pi_i^*(\varphi)$$

Define: $y_i \equiv \pi_i^{*-1}(\kappa_i - \delta \Psi_i)$. This is the type of firm that is out of the market and indifferent about whether to enter.
Because $\pi_i^*(\varphi)$ is an increasing function and $\mu_i \kappa_i - \delta \Psi_i < \kappa_i - \delta \Psi_i$, it must be true that: $x_i < y_i$.

1.4 Weighted Average Productivity

We suppress industry notation. Define the weighted average productivity of firms as:

$$\widetilde{\varphi}_x = \widetilde{\varphi}(x) \equiv \left[\frac{1}{1 - G(x)} \int_x^{\infty} \varphi^{\sigma-1} dG(\varphi)\right]^{\frac{1}{\sigma-1}} \quad \Leftrightarrow \quad \widetilde{\varphi}_x^{\sigma-1}[1 - G(x)] = \int_x^{\infty} \varphi^{\sigma-1} dG(\varphi)$$

1.5 Continuation Values

We suppress industry notation. Define:

$$\alpha \equiv \frac{1}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}$$
$$\gamma_1 \equiv G(x_i)\left\{\mu_i\left[1-\delta G(y_i)\right]-\delta\left[1-G(y_i)\right]\right\}$$
$$\gamma_2 \equiv \left\{\delta\mu_i G(x_i)-\left[1-\delta+\delta G(x_i)\right]\right\}\left[1-G(y_i)\right]$$

The continuation values are:

$$V_i^{in} = \alpha\left\{\gamma_1\kappa_i+\left[1-G(x_i)\right](1-\delta G_y)\,\pi^*(\widetilde{\varphi}_x)+\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_y)\right\}$$

$$V_i^{out} = \alpha\left[\gamma_2\kappa_i+\delta\left[1-G(x_i)\right]\left[1-G(y_i)\right]\pi^*(\widetilde{\varphi}_x)+\left[1-\delta+\delta G(x_i)\right]\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_y)\right]$$

The details:

$$\begin{aligned}
V_i^{in} &= \int_b^{x_i}\left(\mu_i\kappa_i+\delta V_i^{out}\right)dG(\varphi)+\int_{x_i}^{\infty}\left[\pi_i^*(\varphi)+\delta V_i^{in}\right]dG(\varphi)\\
&= \left(\mu_i\kappa_i+\delta V_i^{out}\right)G(x_i)+\delta V_i^{in}\left[1-G(x_i)\right]+\int_{x_i}^{\infty}\pi_i^*(\varphi)\,dG(\varphi)\\
&= \frac{1}{1-\delta\left[1-G(x_i)\right]}\left\{\left(\mu_i\kappa_i+\delta V_i^{out}\right)G(x_i)+\pi^*(\widetilde{\varphi}_{xi})\left[1-G(x_i)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
V_i^{out} &= \int_0^{y_i}\delta V_i^{out}dG(\varphi)+\int_{y_i}^{\infty}\left[\pi_i^*(\varphi)-\kappa_i+\delta V_i^{in}\right]dG(\varphi)\\
&= \delta V_i^{out}G(y_i)+\left(\delta V_i^{in}-\kappa_i\right)\left[1-G(y_i)\right]+\int_{y_i}^{\infty}\pi_i^*(\varphi)\,dG(\varphi)\\
&= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

Substitution:

$$\begin{aligned}
\left\{1-\delta\left[1-G(x_i)\right]\right\}V_i^{in} &= G(x_i)\,\mu_i\kappa_i+\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left(\frac{1-G(y_i)}{1-\delta G(y_i)}\right)\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

$$\begin{aligned}
\left\{1-\delta\left[1-G(x_i)\right]\right\}\left[1-\delta G(y_i)\right]V_i^{in} &= G(x_i)\left[1-\delta G(y_i)\right]\mu_i\kappa_i+\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left[1-G(y_i)\right]\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

$$\begin{aligned}
&V_i^{in}\left\langle\left\{1-\delta\left[1-G(x_i)\right]\right\}\left[1-\delta G(y_i)\right]-\delta^2G(x_i)\left[1-G(y_i)\right]\right\rangle\\
&\quad=\left\{G(x_i)\left[1-\delta G(y_i)\right]\mu_i-\delta G(x_i)(1-G_{yi})\right\}\kappa_i+\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})
\end{aligned}$$

$$
\begin{aligned}
&V_i^{in}(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right] \\
&\quad = \left\{\left[1-\delta G(y_i)\right]\mu_i - \delta\left[1-G(y_i)\right]\right\} G(x_i)\kappa_i + \left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi}) \\
&\quad +\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})
\end{aligned}
$$

$$
\begin{aligned}
V_i^{in} &= \frac{1}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\langle\left\{\left[1-\delta G(y_i)\right]\mu_i-\delta\left[1-G(y_i)\right]\right\}G(x_i)\kappa_i\right. \\
&\quad \left. +\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})+\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})\right\rangle
\end{aligned}
$$

And:

$$
\begin{aligned}
V_i^{out} &= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\frac{\delta\left\{\left[1-\delta G(y_i)\right]\mu_i-\delta\left[1-G(y_i)\right]\right\}G(x_i)\kappa_i+\delta\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})+\delta^2 G(x_i)\left[1-G(\right.}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\right. \\
&= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\frac{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\left[1-\delta G(y_i)\right]\kappa_i+\delta\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})+\left[1-\delta+\delta G(x_i)\right]\left[\right.}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\right. \\
&= \frac{1-G(y_i)}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\kappa_i+\delta\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi})+\left[1-\delta+\delta G(x_i)\right]\pi_i^*(\widetilde{\varphi}_{yi}\right.
\end{aligned}
$$

$$
V_i^{out} = \frac{1-G(y_i)}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\kappa_i+\delta\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi})+\left[1-\delta+\delta G(x_i)\right]\pi_i^*(\widetilde{\varphi}_{yi}\right.
$$

1.6 Free Entry

We suppress industry notation. Free entry (FE) requires that $V_i^{in} = \beta_i^{in}$ and $V_i^{out} = \beta_i^{out}$. So it must be true that $\Psi_i(x,y) = \beta_i^{in} - \beta_i^{out} = \Psi_i$ for $i \in \{d,f\}$.

1.7 Zero Profit Conditions

We suppress industry notation. Because "in" firms are indifferent at cutpoint x_i, and "out" firms are indifferent at cutpoint y_i, we have four zero profit conditions (ZPCs):

$$
\begin{aligned}
\pi_i^*(x_i) &= \mu_i\kappa_i - \delta\Psi_i \\
\pi_i^*(y_i) &= \kappa_i - \delta\Psi_i \qquad \text{for: } i\in\{d,f\}
\end{aligned}
$$

These imply that:

$$
\begin{aligned}
r_i^*(x_i) &= \sigma(\mu_i\kappa_i+\alpha_i) \\
r_i^*(y_i) &= \sigma(\kappa_i+\alpha_i) \qquad \text{for: } i\in\{d,f\} \text{ and } \alpha_i \equiv c-\delta\Psi_i > 0
\end{aligned}
$$

Note that:

$$
\frac{r_d^*(x_d)}{r_d^*(y_d)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1}QP^\sigma}{\left(\frac{y_d}{\rho}\right)^{\sigma-1}QP^\sigma} = \left(\frac{x_d}{y_d}\right)^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d+\alpha_d)}{\sigma(\kappa_d+\alpha_d)} \quad\Leftrightarrow\quad y_d = x_d\left(\frac{\kappa_d+\alpha_d}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}}
$$

Note that: $x_d < y_d \Leftrightarrow \mu_d < 1$, which always holds.

$$
\frac{r_d^*(x_d)}{r_f^*(x_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1}QP^\sigma}{\left[\frac{x_f}{\rho(1+\tau)}\right]^{\sigma-1}QP^\sigma} = \left[\frac{x_d(1+\tau)}{x_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d+\alpha_d)}{\sigma(\mu_f\kappa_f+\alpha_f)} \quad\Leftrightarrow\quad x_f = x_d(1+\tau)\left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}}
$$

Note that: $x_d < x_f \Leftrightarrow \mu_d\kappa_d - \delta\Psi_d + c < (1+\tau_f)^{\sigma-1}(\mu_f\kappa_f - \delta\Psi_f + c)$

$$\frac{r_d^*(x_d)}{r_f^*(y_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left[\frac{y_f}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma} = \left[\frac{x_d(1+\tau)}{y_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\kappa_f + \alpha_f)} \quad \Leftrightarrow \quad y_f = x_d(1+\tau)\left(\frac{\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}$$

Note that $x_f < y_f \Leftrightarrow \mu_f < 1$.

So the cutpoints (y_d, x_f, y_f) can all be expressed in terms of variable x_d:

$$\begin{aligned} y_d(x_d) &= \eta_1 x_d \qquad \text{where: } \eta_1 \equiv \left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\ x_f(x_d) &= \eta_2(1+\tau)x_d \qquad \text{where: } \eta_2 \equiv \left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\ y_f(x_d) &= \eta_3(1+\tau)x_d \qquad \text{where: } \eta_3 \equiv \left(\frac{\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \end{aligned}$$

Recall that we have suppressed the industry subscript. So each industry has a different set of cutpoints.

1.8 Firm Masses

We suppress industry notation. The total mass of i firms is:

$$M_i = M_i^{in} + M_i^{out}$$

Stationarity requires the following:

$$\begin{aligned} M_i^{in} &= [1-G(x_i)]M_i^{in} + [1-G(y_i)]M_i^{out} \\ &\Leftrightarrow \quad M_i^{in} = \left(\frac{1-G(y_i)}{G(x_i)}\right)M_i^{out} \\ &\Leftrightarrow \quad M_i^{in} = \left(\frac{1-G(y_i)}{G(x_i)}\right)(M_i - M_i^{in}) \\ &\Leftrightarrow \quad M_i^{in}\left(\frac{G(x_i)+1-G(y_i)}{G(x_i)}\right) = \left(\frac{1-G(y_i)}{G(x_i)}\right)M_i \\ &\Leftrightarrow \quad M_i^{in} = \left(\frac{1-G(y_i)}{G(x_i)+1-G(y_i)}\right)M_i \end{aligned}$$

$$\begin{aligned} M_i^{out} &= G(x_i)M_i^{in} + G(y_i)M_i^{out} \\ &\Leftrightarrow \quad M_i^{out} = \left(\frac{G(x_i)}{1-G(y_i)}\right)M_i^{in} = \left(\frac{G(x_i)}{1-G(y_i)}\right)\left(\frac{1-G(y_i)}{G(x_i)+1-G(y_i)}\right)M_i \\ &\Leftrightarrow \quad M_i^{out} = \left(\frac{G(x_i)}{G(x_i)+1-G(y_i)}\right)M_i \end{aligned}$$

1.9 Labor Market Clearing

We suppress industry notation. Note that the distribution of new producers (firms that were "out", but then entered) differs from the distribution of old producers (firms that were "in" and then stayed). Namely:

$$h_i^{old} = \begin{cases} \frac{g(\varphi)}{1-G(x_i)} & \text{if } x_i \le \varphi \\ 0 & \text{if } \varphi < x_i \end{cases} \quad \text{and} \quad h_i^{new} = \begin{cases} \frac{g(\varphi)}{1-G(y_i)} & \text{if } y_i \le \varphi \\ 0 & \text{if } \varphi < y_i \end{cases}$$

Recall that:

$$\widetilde{\varphi}_x = \widetilde{\varphi}(x) \equiv \left[\frac{1}{1-G(x)} \int_x^\infty \varphi^{\sigma-1} dG(\varphi)\right]^{\frac{1}{\sigma-1}} \quad \Leftrightarrow \quad \widetilde{\varphi}_x^{\sigma-1}\left[1-G(x)\right] = \int_x^\infty \varphi^{\sigma-1} dG(\varphi)$$

Aggregate profit for i firms in a specific period t is:

$$\begin{aligned}
\Pi_i^t &= \underbrace{\left[1-G(x_i)\right] M_i^{in}}_{\text{mass of old firms (that stay)}} \underbrace{\int_{x_i}^\infty \pi_i^*(\varphi)\, h_i^{old}(\varphi)\, d\varphi}_{\text{old firm profits}} + \underbrace{\left[1-G(y_i)\right] M_i^{out}}_{\text{mass of new firms}} \underbrace{\int_{y_i}^\infty \pi_i^*(\varphi)\, h_i^{new}(\varphi)\, d\varphi}_{\text{new firm profits}} \\
&= M_i^{in} \int_{x_i}^\infty \left(\phi_i \varphi^{\sigma-1} - c\right) dG(\varphi) + M_i^{out} \int_{y_i}^\infty \left(\phi_i \varphi^{\sigma-1} - c\right) dG(\varphi) \\
&= \phi_i M_i^{in} \int_{x_i}^\infty \varphi^{\sigma-1} g(\varphi)\, d\varphi - c M_i^{in}\left[1-G(x_i)\right] + \phi_i M_i^{out} \int_{y_i}^\infty \varphi^{\sigma-1} g(\varphi)\, d\varphi - c M_i^{out}\left[1-G(y_i)\right] \\
&= \phi_i M_i^{in} \widetilde{\varphi}_{xi}^{\sigma-1}\left[1-G(x_i)\right] - c M_i^{in}\left[1-G(x_i)\right] + \phi_i M_i^{out} \widetilde{\varphi}_{yi}^{\sigma-1}\left[1-G(y_i)\right] - c M_i^{out}\left[1-G(y_i)\right] \\
&= M_i^{in}\left(\phi_i \widetilde{\varphi}_{xi}^{\sigma-1} - c\right)\left[1-G(x_i)\right] + M_i^{out}\left(\phi_i \widetilde{\varphi}_{yi}^{\sigma-1} - c\right)\left[1-G(y_i)\right] \\
&= M_i^{in} \pi_i^*(\widetilde{\varphi}_{xi})\left[1-G(x_i)\right] + M_i^{out} \pi_i^*(\widetilde{\varphi}_{yi})\left[1-G(y_i)\right]
\end{aligned}$$

The present value of aggregate profit for i firms over time is therefore:

$$\Pi_i = M_i^{in} \pi_i^*(\widetilde{\varphi}_{xi})\left[1-G(x_i)\right] + M_i^{out} \pi_i^*(\widetilde{\varphi}_{yi})\left[1-G(y_i)\right] + \delta M_i^{in} V_i^{in} + \delta M_i^{out} V_i^{out}$$

We can now invoke the free entry conditions:

$$\beta_i^{in} = V_i^{in} = \alpha\left\{\gamma_1 \kappa_i + \left[1-G(x_i)\right]\left[1-\delta G(y_i)\right] \pi^*(\widetilde{\varphi}_{xi}) + \delta G(x_i)\left[1-G(y_i)\right] \pi_i^*(\widetilde{\varphi}_{yi})\right\}$$

$$\beta_i^{out} = V_i^{out} = \alpha\left\{\gamma_2 \kappa_i + \delta\left[1-G(x_i)\right]\left[1-G(y_i)\right] \pi^*(\widetilde{\varphi}_{xi}) + \left[1-\delta+\delta G(x_i)\right]\left[1-G(y_i)\right] \pi_i^*(\widetilde{\varphi}_{yi})\right\}$$

By manipulating these conditions, we can isolate $\pi^*(\widetilde{\varphi}_{xi})$ and $\pi^*(\widetilde{\varphi}_{yi})$ to show that:

$$\pi^*(\widetilde{\varphi}_{xi})\left[1-G(x_i)\right] = \left[1-\delta+\delta G(x_i)\right]\beta_i^{in} - \delta G(x_i)\,\beta_i^{out} - G(x_i)\,\mu_i \kappa_i \qquad (\star)$$

$$\pi_i^*(\widetilde{\varphi}_{yi})\left[1-G(y_i)\right] = \left[1-\delta G(y_i)\right]\beta_i^{out} - \delta\left[1-G(y_i)\right]\beta_i^{in} + \left[1-G(y_i)\right]\kappa_i$$

Substitution yields:

$$\begin{aligned}
\Pi_i &= M_i^{in}\left\{\left[1-\delta+\delta G(x_i)\right]\beta_i^{in} - \delta G(x_i)\,\beta_i^{out} - G(x_i)\,\mu_i \kappa_i\right\} \\
&\quad + M_i^{out}\left\{\left[1-\delta G(y_i)\right]\beta_i^{out} - \delta\left[1-G(y_i)\right]\beta_i^{in} + \left[1-G(y_i)\right]\kappa_i\right\} + \delta M_i^{in}\beta_i^{in} + \delta M_i^{out}\beta_i^{out}
\end{aligned}$$

$$\begin{aligned}
\Pi_i &= \left\{\left[1+\delta G(x_i)\right] M_i^{in} - \delta\left[1-G(y_i)\right] M_i^{out}\right\}\beta_i^{in} \\
&\quad + \left\{\left[1+\delta-\delta G(y_i)\right] M_i^{out} - \delta G(x_i)\, M_i^{in}\right\}\beta_i^{out} + \left[1-G(y_i)\right]\kappa_i M_i^{out} - G(x_i)\,\mu_i \kappa_i M_i^{in}
\end{aligned}$$

$$\begin{aligned}\Pi_i &= \left\{[1+\delta G(x_i)]\,M_i^{in} - \delta\,[1-G(y_i)]\,M_i^{out}\right\}\beta_i^{in} + [1-G(y_i)]\,M_i^{out}\kappa_i - G_x M_i^{in}\mu_i\kappa_i \\ &\quad + \left\{[1+\delta-\delta G(y_i)]\,M_i^{out} - \delta G(x_i)\,M_i^{in}\right\}\beta_i^{out}\end{aligned}$$

Using the equilibrium masses yields:

$$\begin{aligned}\Pi_i &= \frac{M_i}{G(x_i)+1-G(y_i)}\left\{[1+\delta G(x_i)]\,[1-G(y_i)] - \delta\,[1-G(y_i)]\,G(x_i)\right\}\beta_i^{in} + [1-G(y_i)]\,M_i^{out}\kappa_i - G(x_i)\,M_i^{in}\mu_i\kappa_i \\ &\quad + \frac{M_i}{G(x_i)+1-G(y_i)}\left\{[1+\delta-\delta G(y_i)]\,G(x_i) - \delta G(x_i)\,[1-G(y_i)]\right\}\beta_i^{out}\end{aligned}$$

$$\begin{aligned}\Pi_i &= \frac{M_i\,[1-G(y_i)]}{G(x_i)+1-G(y_i)}\beta_i^{in} + \frac{M_i G(x_i)}{G(x_i)+1-G(y_i)}\beta_i^{out} + [1-G(y_i)]\,M_i^{out}\kappa_i - G(x_i)\,M_i^{in}\mu_i\kappa_i \\ &= \underbrace{M_i^{in}\beta_i^{in} + M_i^{out}\beta_i^{out}}_{L_{ti}} + \underbrace{[1-G(y_i)]\,M_i^{out}\kappa_i}_{L_{si}} - \underbrace{G(x_i)\,M_i^{in}\mu_i\kappa_i}_{L_{ri}}\end{aligned}$$

where:

- L_{ti} is labor spent on learning each firm's type
- L_{si} is labor that "out" firms spend on setting up new production when they enter the market
- L_{ri} is labor that is recovered when "in" firms decide to exit the market

Total revenue by i firms is the total profits, plus the total production costs, L_{pi}. Then:

$$R_i = \Pi_i + L_{pi} = L_{pi} + L_{ti} + L_{si} - L_{ri} = L_i$$

So the labor market for a given industry clears.

Note that given the equilibrium demand function:

$$\begin{aligned}\int\limits_{v\in V_j} p_j(v)\,q_j(v)\,dv &= \int\limits_{v\in V_j} p_j(v)^{1-\sigma}\,Q_j P_j^{\sigma}\,dv \\ &= Q_j P_j^{\sigma}\int\limits_{v\in V_j} p_j(v)^{1-\sigma}\,dv \\ &= Q_j P_j^{\sigma} P_j^{1-\sigma} = P_j Q_j\end{aligned}$$

So by going back to our original Lagrangian, we can see that:

$$\begin{aligned}\mathcal{L} &= \left(1-\sum_{j=1}^{J} w_j\right)\log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda\left[R - \sum_{j=0}^{J}\int\limits_{v\in V_j} p_j(v)\,q_j(v)\,dv\right] \\ &= \left(1-\sum_{j=1}^{J} w_j\right)\log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda\left[R - \sum_{j=0}^{J} P_j Q_j\right]\end{aligned}$$

Then for $j \in \{1,\ldots,J\}$, the FOC becomes:

$$\frac{\partial \mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \lambda P_j = 0 \quad \Leftrightarrow \quad \lambda = \frac{w_j}{P_j Q_j}$$

And:

$$1 = \sum_{j=0}^{J} w_j = \lambda \sum_{j=0}^{J} P_j Q_j = \lambda \sum_{j=0}^{J} R_j = \lambda R = \lambda L \quad \Leftrightarrow \quad \lambda = \frac{1}{L}$$

So:

$$\frac{\partial \mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \frac{P_j}{L} = 0 \quad \Leftrightarrow \quad L w_j = P_j Q_j = R_j = L_i$$

So the labor market across all industries clears.

1.10 Equilibrium Characterization under the Pareto Distribution

We suppress industry notation. Suppose that types are chosen according to the Pareto distribution, iid over time and players, with domain $\varphi \sim [b, \infty)$ for small $b > 0$.

$$\begin{aligned} \text{Density function:} \quad g(\varphi) &= \frac{ab^a}{\varphi^{a+1}} \quad \text{for: } \varphi \geq b \\ \text{Distribution function:} \quad G(y) &= \Pr(\varphi \leq y) = 1 - \left(\frac{b}{y}\right)^a \quad \text{for: } x \geq b \end{aligned}$$

Equilibrium behavior is defined by the function:

$$\pi_d^*(x_d) = \phi_d x_d^{\sigma-1} - c = \mu_d \kappa_d - \delta \Psi_d$$

where:

$$\phi_d = \frac{QP^\sigma}{\sigma \rho^{\sigma-1}} = \frac{RP^{\sigma-1}}{\sigma \rho^{\sigma-1}} = \frac{LP^{\sigma-1}}{\sigma \rho^{\sigma-1}}$$

Define $z \equiv a - \sigma + 1$. Under the Pareto distribution, for $x \geq b$:

$$\begin{aligned} \int_x^\infty \varphi^{\sigma-1} dG(\varphi) &= \int_x^\infty \varphi^{\sigma-1} \left(\frac{ab^a}{\varphi^{a+1}}\right) d\varphi = ab^a \int_x^\infty \varphi^{\sigma-a-2} d\varphi \\ &= ab^a \lim_{t\to\infty} \int_x^t \varphi^{\sigma-a-2} d\varphi = ab^a \lim_{t\to\infty} \left[\frac{\varphi^{\sigma-a-1}}{\sigma-a-1}\right]_x^t = -\frac{ab^a}{z} \lim_{t\to\infty} \left[\frac{1}{t^z} - \frac{1}{x^z}\right] \\ &= \frac{ab^a}{z}\left(\frac{1}{x^z}\right) = \frac{ab^a}{z} x^{-z} \end{aligned}$$

The equilibrium price index is:

$$\begin{aligned} P(x_d) &= \rho \left[\int_{x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + \int_{\eta_1 x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + (1+\tau)^{1-\sigma} \left(\int_{\eta_2(1+\tau)x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + \int_{\eta_3(1+\tau)x_d}^\infty \varphi^{\sigma-1} dG(\varphi)\right)\right]^{\frac{1}{1-\sigma}} \\ &= \rho \left\{\frac{ab^a}{z} x_d^{-z} + \frac{ab^a}{z}(\eta_1 x_d)^{-z} + (1+\tau)^{1-\sigma}\left[\frac{ab^a}{z}(\eta_2(1+\tau)x_d)^{-z} + \frac{ab^a}{z}(\eta_3(1+\tau)x_d)^{-z}\right]\right\}^{\frac{1}{1-\sigma}} \\ &= \rho \left(\frac{z x_d^z}{ab^a \Phi}\right)^{\frac{1}{\sigma-1}} \qquad \text{where: } \Phi \equiv 1 + \eta_1^{-z} + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right) \end{aligned}$$

And:

$$P\left(x_d\right)^{\sigma-1}=\frac{\rho^{\sigma-1}zx_d^z}{ab^a\Phi}$$

So the best response function is:

$$\begin{aligned}
\Rightarrow \pi_d^*\left(x_d\right)=\frac{LP^{\sigma-1}}{\sigma\rho^{\sigma-1}}x_d^{\sigma-1}-c=\mu_d\kappa_d-\delta\Psi_d \\
\Leftrightarrow \frac{L}{\sigma\rho^{\sigma-1}}\left(\frac{\rho^{\sigma-1}zx_d^z}{ab^a\Phi}\right)x_d^{\sigma-1}=\mu_d\kappa_d+\alpha_d \\
\Leftrightarrow x_d^*\left(\tau\right)=\psi\Phi^{\frac{1}{a}} \quad \text{where: } \psi\equiv\left[\frac{\sigma ab^a}{Lz}\left(\mu_d\kappa_d+\alpha_d\right)\right]^{\frac{1}{a}}
\end{aligned}$$

QED.

Intermediate Calculations for Comparative Statics

Cutpoint scalars

$$\begin{aligned}
\eta_1 &= \left(\frac{\kappa_d+\alpha_d}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_1}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_d+\alpha_d}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\left(\mu_d\kappa_d+\alpha_d\right)-\mu_d\left(\kappa_d+\alpha_d\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{\left(1-\mu_d\right)\alpha_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)\left(\kappa_d+\alpha_d\right)} \\
\frac{\partial\eta_1}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_d+\alpha_d}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\kappa_d+\alpha_d\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\kappa_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}
\end{aligned}$$

$$\begin{aligned}
\eta_2 &= \left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_2}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\mu_d\left(\mu_f\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\mu_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)} \\
\frac{\partial\eta_2}{\partial\kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\mu_f}{\mu_d\kappa_d+\alpha_d}\right)=\frac{\mu_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f+\alpha_f\right)} \\
\frac{\partial\eta_2}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\mu_f\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\kappa_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)} \\
\frac{\partial\eta_2}{\partial\mu_f} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\kappa_f}{\mu_d\kappa_d+\alpha_d}\right)=\frac{\kappa_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f+\alpha_f\right)}
\end{aligned}$$

$$
\begin{aligned}
\eta_3 &= \left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_3}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\mu_d\left(\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\mu_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)} \\
\frac{\partial\eta_3}{\partial\kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{1}{\mu_d\kappa_d+\alpha_d}\right)=\frac{\eta_3}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)} \\
\frac{\partial\eta_3}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\kappa_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}
\end{aligned}
$$

<u>Φ partials</u>

$$
\Phi\left(\tau\right)\equiv 1+\eta_1^{-z}+\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial\tau}=-a\left(1+\tau\right)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial y}=-z\left[\eta_1^{-z-1}\frac{\partial\eta_1}{\partial y}+\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\frac{\partial\eta_2}{\partial y}+\eta_3^{-z-1}\frac{\partial\eta_3}{\partial y}\right)\right]\text{ for: } y=\kappa_f,\kappa_d,\mu_f,\mu_d
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\kappa_f} &= -z\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\left[\frac{\mu_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f+\alpha_f\right)}\right]+\eta_3^{-z-1}\left[\frac{\eta_3}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\right]\right) \\
&= \frac{-z}{\left(\sigma-1\right)\left(1+\tau\right)^a}\left(\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\kappa_d} &= -z\left\{\eta_1^{-z-1}\left[\frac{\left(1-\mu_d\right)\alpha_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)\left(\kappa_d+\alpha_d\right)}\right]+\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\left[\frac{-\mu_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\right]+\eta_3^{-z-1}\left[\frac{-\mu_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\right.}\right.\right.\right. \\
&= \frac{-z}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\left[\frac{\left(1-\mu_d\right)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d}-\frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\left(1+\tau\right)^a}\right]
\end{aligned}
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial\mu_f}=\frac{-z\kappa_f\eta_2^{-z}}{\left(\sigma-1\right)\left(1+\tau\right)^a\left(\mu_f\kappa_f+\alpha_f\right)}
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\mu_d} &= -z\left[\frac{-\kappa_d\eta_1^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}+\left(1+\tau\right)^{-a}\left(\frac{-\kappa_d\eta_2^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}+\frac{-\kappa_d\eta_3^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\right)\right] \\
&= \frac{z\kappa_d}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\left[\eta_1^{-z}+\frac{\eta_2^{-z}+\eta_3^{-z}}{\left(1+\tau\right)^{-a}}\right]
\end{aligned}
$$

Cutpoint x_d

$$\frac{\partial x_d^*}{\partial \tau} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \tau} = \frac{-\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d}{(1+\tau)^{a+1}\Phi}$$

$$\frac{\partial x_d^*}{\partial \kappa_f} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \kappa_f} = \frac{x_d}{a\Phi}\left(\frac{\partial \Phi}{\partial \kappa_f}\right) = \frac{-z x_d}{a(\sigma-1)(1+\tau)^a\Phi}\left(\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)$$

$$\frac{\partial x_d^*}{\partial \mu_f} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \mu_f} = \frac{-z\kappa_f\eta_2^{-z}x_d}{a(\sigma-1)(1+\tau)^a(\mu_f\kappa_f+\alpha_f)\Phi}$$

$$\begin{aligned}
\frac{\partial x_d^*}{\partial \kappa_d} &= \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \kappa_d} + \frac{\partial \psi}{\partial \kappa_d}\Phi^{\frac{1}{a}} \\
&= \frac{-z x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d} - \frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right] + \frac{1}{a}\left[\frac{\sigma a b^a}{Lz}(\mu_d\kappa_d+\alpha_d)\right]^{\frac{1}{a}-1}\left(\frac{\sigma a b^a\mu_d}{Lz}\right)\Phi^{\frac{1}{a}} \\
&= \frac{-z x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d} - \frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right] + \frac{x_d}{a}\left(\frac{\mu_d}{\mu_d\kappa_d+\alpha_d}\right) \\
&= \frac{x_d}{a(\mu_d\kappa_d+\alpha_d)}\left\{\mu_d - \frac{z}{(\sigma-1)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d} - \frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_d^*}{\partial \mu_d} &= \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \mu_d} + \frac{\partial \psi}{\partial \mu_d}\Phi^{\frac{1}{a}} \\
&= \frac{z\kappa_d x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\eta_1^{-z} + \frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right] + \frac{1}{a}\left[\frac{\sigma a b^a}{Lz}(\mu_d\kappa_d+\alpha_d)\right]^{\frac{1}{a}-1}\left(\frac{\sigma a b^a\kappa_d}{Lz}\right)\Phi^{\frac{1}{a}} \\
&= \frac{z\kappa_d x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\eta_1^{-z} + \frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right] + \frac{x_d}{a}\left(\frac{\kappa_d}{\mu_d\kappa_d+\alpha_d}\right) \\
&= \frac{\kappa_d x_d}{a(\mu_d\kappa_d+\alpha_d)}\left\{1 + \frac{z}{(\sigma-1)\Phi}\left[\eta_1^{-z} + \frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right]\right\}
\end{aligned}$$

Proposition 2. Higher government takings from foreign firms are associated with higher foreign productivity and lower domestic productivity.

Proof of Proposition 2.

For $i \in \{d, f\}$, the average weighted productivity of firms that are producing in the economy (and hence observed) is:

$$
\begin{aligned}
\widetilde{\varphi}_i^{in} &= \left[\frac{1}{1-G(x_i)}\int\limits_{x_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]^{\frac{1}{\sigma-1}} \\
\widetilde{\varphi}_i^{out} &= \left[\frac{1}{1-G(y_i)}\int\limits_{y_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]^{\frac{1}{\sigma-1}} \\
\widetilde{\varphi}_i &= \left\{\frac{1}{M_i^t}\left[M_i^{in}\left(\widetilde{\varphi}_i^{in}\right)^{\sigma-1}+M_i^{out}\left(\widetilde{\varphi}_i^{out}\right)^{\sigma-1}\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{1}{M_i^t}\left[M_i^{in}\frac{1}{1-G(x_i)}\int\limits_{x_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi+M_i^{out}\frac{1}{1-G(y_i)}\int\limits_{y_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{1}{M_i^t}\left[M_i^{in}\int\limits_{x_i}^{\infty}\varphi^{\sigma-1}h_i^{in}(\varphi)\,d\varphi+M_i^{out}\int\limits_{y_i}^{\infty}\varphi^{\sigma-1}h_i^{out}(\varphi)\,d\varphi\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{1}{1-G(x_i)+1-G(y_i)}\left[\int\limits_{x_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi+\int\limits_{y_i}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]\right\}^{\frac{1}{\sigma-1}}
\end{aligned}
$$

Under the Pareto distribution:

$$
\begin{aligned}
\widetilde{\varphi}_i &= \left\{\frac{1}{\left(\frac{b}{x_i}\right)^a+\left(\frac{b}{y_i}\right)^a}\left[\int\limits_{x_i}^{\infty}\varphi^{\sigma-1}\left(\frac{ab^a}{\varphi^{a+1}}\right)d\varphi+\int\limits_{y_i}^{\infty}\varphi^{\sigma-1}\left(\frac{ab^a}{\varphi^{a+1}}\right)d\varphi\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{ab^a}{b^a\left(x_i^{-a}+y_i^{-a}\right)}\left[\int\limits_{x_i}^{\infty}\varphi^{-z-1}d\varphi+\int\limits_{y_i}^{\infty}\varphi^{-z-1}d\varphi\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{a}{\left(x_i^{-a}+y_i^{-a}\right)}\left[\lim_{t\to\infty}\int\limits_{x_i}^{t}\varphi^{-z-1}d\varphi+\lim_{t\to\infty}\int\limits_{y_i}^{t}\varphi^{-z-1}d\varphi\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{a}{\left(x_i^{-a}+y_i^{-a}\right)}\left(\lim_{t\to\infty}\left[\frac{\varphi^{-z}}{-z}\right]_{x_i}^{t}+\lim_{t\to\infty}\left[\frac{\varphi^{-z}}{-z}\right]_{y_i}^{t}\right)\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{a}{-z\left(x_i^{-a}+y_i^{-a}\right)}\left(\lim_{t\to\infty}\left[\frac{1}{\varphi^{z}}\right]_{x_i}^{t}+\lim_{t\to\infty}\left[\frac{1}{\varphi^{z}}\right]_{y_i}^{t}\right)\right\}^{\frac{1}{\sigma-1}} \\
&= \left\{\frac{a}{-z\left(x_i^{-a}+y_i^{-a}\right)}\left[\lim_{t\to\infty}\left(\frac{1}{t^z}-\frac{1}{x_i^z}\right)+\lim_{t\to\infty}\left(\frac{1}{t^z}-\frac{1}{y_i^z}\right)\right]\right\}^{\frac{1}{\sigma-1}} \\
&= \left[\frac{a\left(x_i^{-z}+y_i^{-z}\right)}{z\left(x_i^{-a}+y_i^{-a}\right)}\right]^{\frac{1}{\sigma-1}}
\end{aligned}
$$

For foreign firms:

$$
\begin{aligned}
\widetilde{\varphi}_f &= \left[\frac{a\left(x_f^{-z}+y_f^{-z}\right)}{z\left(x_f^{-a}+y_f^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(\left[\eta_2\left(1+\tau\right)x_d\right]^{-z}+\left[\eta_3\left(1+\tau\right)x_d\right]^{-z}\right)}{z\left(\left[\eta_2\left(1+\tau\right)x_d\right]^{-a}+\left[\eta_3\left(1+\tau\right)x_d\right]^{-a}\right)}\right]^{\frac{1}{\sigma-1}} \\
&= \left[\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)\left(1+\tau\right)^{-z}x_d^{-z}}{z\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(1+\tau\right)^{-a}x_d^{-a}}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)\left(1+\tau\right)^{\sigma-1}x_d^{\sigma-1}}{z\left(\eta_2^{-a}+\eta_3^{-a}\right)}\right]^{\frac{1}{\sigma-1}} \\
&= \nu\left(1+\tau\right)x_d H \quad \text{where: } \nu \equiv \left(\frac{a}{z}\right)^{\frac{1}{\sigma-1}} \text{ and } H \equiv \left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}}
\end{aligned}
$$

$$
\frac{\partial\widetilde{\varphi}_f}{\partial\tau} = \nu H\left[\left(1+\tau\right)\frac{\partial x_d}{\partial\tau}+x_d\right] = \frac{\nu H x_d}{a\Phi}\left[\left(1+\tau\right)\frac{\partial\Phi}{\partial\tau}+a\Phi\right] = \frac{\nu H x_d}{\Phi}\left[\Phi-\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{\nu\left(1+\eta_1^{-z}\right)Hx_d}{\Phi} >
$$

For domestic firms:

$$
\widetilde{\varphi}_d = \left[\frac{a\left(x_d^{-z}+y_d^{-z}\right)}{z\left(x_d^{-a}+y_d^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(x_d^{-z}+\left(\eta_1 x_d\right)^{-z}\right)}{z\left(x_d^{-a}+\left(\eta_1 x_d\right)^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(1+\eta_1^{-z}\right)x_d^{-z}}{z\left(1+\eta_1^{-a}\right)x_d^{-a}}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(1+\eta_1^{-z}\right)x_d^{\sigma-1}}{z\left(1+\eta_1^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(1+\right.}{z\left(1+\right.}\right.
$$

$$
\frac{\partial\widetilde{\varphi}_d}{\partial\tau} = \left[\frac{a\left(1+\eta_1^{-z}\right)}{z\left(1+\eta_1^{-a}\right)}\right]^{\frac{1}{\sigma-1}}\frac{\partial x_d}{\partial\tau} < 0
$$

QED.

Proposition 3. There exists a political equilibrium in which the host government chooses an optimal takings rate, and firms operate in the resulting market equilibrium.

Proof of Proposition 3.

Suppressing industry subscripts, the government's objective function is:

$$
U\left(\tau\right) = \tau Q_f\left(\tau\right)
$$

The first order condition is thus:

$$
\begin{aligned}
\frac{\partial U\left(\tau\right)}{\partial\tau} &= \tau\frac{\partial Q_f\left(\tau\right)}{\partial\tau}+Q_f\left(\tau\right) \\
&= -\frac{\tau Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)\psi}{\rho\left(z-1\right)\left(1+\tau\right)^{a+1}}\Phi^{\frac{1-2a}{a}}\left[a\left(1+\eta_1^{-z}\right)+\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]+\frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)\psi}{\rho\left(z-1\right)\left(1+\tau\right)^{a}}\Phi^{\frac{1-a}{a}} \\
&= \frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)\psi}{\rho\left(z-1\right)\left(1+\tau\right)^{a+1}}\Phi^{\frac{1-2a}{a}}\left\{\left(1+\tau\right)\Phi-\tau\left[a\left(1+\eta_1^{-z}\right)+\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]\right\} \\
&= \frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)\psi}{\rho\left(z-1\right)\left(1+\tau\right)^{a+1}}\Phi^{\frac{1-2a}{a}}\left[\left(1+\tau-\tau a\right)\left(1+\eta_1^{-z}\right)+\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]
\end{aligned}
$$

Note that:

$$\frac{\partial U}{\partial \tau} \geq 0 \Leftrightarrow 0 \leq (1+\tau-\tau a)\left(1+\eta_1^{-z}\right)+(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right) \equiv \Gamma(\tau)$$

Additionally, note that:

$$\Gamma(\tau=0)=1+\eta_1^{-z}+\eta_2^{-z}+\eta_3^{-z}>0$$

$$\frac{\partial \Gamma(\tau)}{\partial \tau}=-(a-1)\left(1+\eta_1^{-z}\right)-a(1+\tau)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)<0$$

$$\lim_{\tau \rightarrow \infty} \Gamma(\tau)=\lim_{\tau \rightarrow \infty}\left\{[1-\tau(a-1)]\left(1+\eta_1^{-z}\right)+\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^a}\right\}=-\infty<0$$

So by the intermediate value theorem, there is a unique optimal takings rate. Furthermore:

$$\frac{\partial U(\tau)}{\partial \tau}=0 \Leftrightarrow \Gamma(\tau)=0$$

So the political equilibrium for a given industry is characterized by the implicitly defined variable τ_j^*, where τ_j^* solves:

$$\Gamma_j\left(\tau_j^*\right)=\left(1+\tau_j^*-\tau_j^* a\right)\left(1+\eta_{j1}^{-z}\right)+\left(1+\tau_j^*\right)^{-a}\left(\eta_{j2}^{-z}+\eta_{j3}^{-z}\right)=0$$

This solution in turn determines the equilibrium cutpoints, $\left(x_d^*, y_d^*, x_f^*, y_f^*\right)$. QED.

Lemma. In equilibrium:

$$1<\tau^*(a-1)$$

<u>Proof of Lemma.</u>

$$\Gamma\left(\tau^*\right)=0 \quad \Leftrightarrow \quad 1+\tau^*-\tau^* a=-\frac{\eta_2^{-z}+\eta_3^{-z}}{\left(1+\tau^*\right)^a\left(1+\eta_1^{-z}\right)}<0 \quad \Leftrightarrow \quad 1<\tau^*(a-1)$$

QED.

Proposition 4. For foreign firms, higher startup costs and higher mobility are associated with lower government takings.

Proof of Proposition 4.
Suppressing industry subscripts, recall that the equilibrium taking rate, τ, solves:

$$\Gamma(\tau^*) = (1+\tau^*-\tau^* a)\left(1+\eta_1^{-z}\right) + (1+\tau^*)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right) = 0$$

So by the implicit function theorem:

$$\frac{\partial \tau^*}{\partial y} = \frac{-\Gamma_y}{\Gamma_{\tau^*}} \quad \text{for any exogenous parameter } y$$

Note that:

$$\Gamma_{\tau^*} = -\left[(a-1)\left(1+\eta_1^{-z}\right) + a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] < 0$$

Foreign startup costs

$$\begin{aligned}
\Gamma_{\kappa_f} &= -\frac{z}{(\sigma-1)(1+\tau)^a}\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f} + \frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right) < 0 \\
\Rightarrow \frac{\partial\tau^*}{\partial\kappa_f} &= \frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}} = \frac{-z\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)}{(\sigma-1)(1+\tau)^a\left[(a-1)\left(1+\eta_1^{-z}\right)+a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]} < 0
\end{aligned}$$

Foreign mobility

$$\Gamma_{\mu_f} = -z(1+\tau^*)^{-a}\eta_2^{-z-1}\frac{\partial\eta_2}{\partial\mu_f} < 0$$

QED.

Proposition 5. For foreign firms, higher startup costs are associated with higher productivity when foreign asset mobility is high.

Proof of Proposition 5.
(a) As shown in the Proof for Proposition 2, the average observed weighted productivity for foreign firms is:

$$\widetilde{\varphi}_f = \nu(1+\tau)x_d H \quad \text{where: } \nu \equiv \left(\frac{a}{z}\right)^{\frac{1}{\sigma-1}} \text{ and } H \equiv \left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}}$$

Define:

$$\begin{aligned}
\phi(-z) \equiv \frac{\partial\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\partial\kappa_f} &= -z\eta_2^{-z-1}\frac{\partial\eta_2}{\partial\kappa_f} - z\eta_3^{-z-1}\frac{\partial\eta_3}{\partial\kappa_f} = -\left(\frac{z}{\sigma-1}\right)\left[\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right] < 0 \\
\Rightarrow \lim_{\mu_f\to 1}\phi(-z) &= -\frac{z\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(\sigma-1)(\kappa_f+\alpha_f)} = -\frac{2z\widehat{\eta}^{-z}}{(\sigma-1)(\kappa_f+\alpha_f)} < 0
\end{aligned}$$

The total effect of foreign start-up costs on $\overline{\varphi}_f$ is:

$$\frac{d\widetilde{\varphi}_f}{d\kappa_f} = \frac{\partial\widetilde{\varphi}_f}{\partial\kappa_f} + \left(\frac{\partial\widetilde{\varphi}_f}{\partial\tau^*}\right)\left(\frac{\partial\tau^*}{\partial\kappa_f}\right)$$

where:

$$\frac{\partial\widetilde{\varphi}_f}{\partial\kappa_f} = \nu\left(1+\tau\right)\left[x_d\frac{\partial H}{\partial\kappa_f} + \frac{\partial x_d}{\partial\kappa_f}H\right] = \nu\left(1+\tau\right)x_d^*\left[\frac{\partial H}{\partial\kappa_f} + \frac{\phi\left(-z\right)}{a\left(1+\tau^*\right)^a\Phi^*}H\right] = \frac{\nu\left(1+\tau\right)x_d^*}{a\left(1+\tau^*\right)^a\Phi^*}\left[a\left(1+\tau^*\right)^a\Phi^*\frac{\partial H}{\partial\kappa_f} + \phi\left(-z\right)\right.$$

$$\begin{aligned}\frac{\partial H}{\partial\kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi\left(-z\right)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi\left(-a\right)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)^2}\right]\\
&= \frac{H}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi\left(-z\right)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi\left(-a\right)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right]\end{aligned}$$

$$\frac{\partial\widetilde{\varphi}_f}{\partial\tau^*} = \nu H\left[\left(1+\tau^*\right)\frac{\partial x_d^*}{\partial\tau^*} + x_d^*\right] = \frac{\nu H x_d^*}{\left(1+\tau^*\right)^a\Phi^*}\left[\left(1+\tau^*\right)^a\Phi^* - \left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{\nu H\left(1+\eta_1^{-z}\right)x_d^*}{\Phi^*} > 0$$

$$\frac{\partial\tau^*}{\partial\kappa_f} = \frac{-\phi\left(-z\right)}{\left(1+\tau^*\right)^a\Gamma_{\tau^*}} < 0$$

Combining this information to calculate the total effect of foreign startup costs yields:

$$\begin{aligned}\frac{d\widetilde{\varphi}_f}{d\kappa_f} &= \frac{\nu\left(1+\tau\right)x_d^*}{a\left(1+\tau^*\right)^a\Phi^*}\left[a\left(1+\tau^*\right)^a\Phi^*\frac{\partial H}{\partial\kappa_f} + \phi\left(-z\right)H\right] - \frac{\nu H\left(1+\eta_1^{-z}\right)\phi\left(-z\right)x_d^*}{\left(1+\tau^*\right)^a\Gamma_{\tau^*}\Phi^*}\\
&= \frac{\nu x_d^*}{a\left(1+\tau^*\right)^{a-1}\Phi^*}\left\{a\left(1+\tau^*\right)^a\Phi^*\frac{\partial H}{\partial\kappa_f} + \phi\left(-z\right)H - \frac{aH\left(1+\eta_1^{-z}\right)\phi\left(-z\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}}\right\}\\
&= \frac{\nu x_d^* H}{a\left(1+\tau^*\right)^{a-1}\Phi^*}\left\{\frac{a\left(1+\tau^*\right)^a\Phi^*}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi\left(-z\right)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi\left(-a\right)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right] + \phi\left(-z\right) - \frac{a\left(1+\eta_1^{-z}\right)\phi\left(-z\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}}\right\}\end{aligned}$$

So:

$$0 < \frac{d\widetilde{\varphi}_f}{d\kappa_f} \quad\Leftrightarrow\quad 0 < \frac{a\left(1+\tau^*\right)^a\Phi^*}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi\left(-z\right)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi\left(-a\right)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right] + \phi\left(-z\right) - \frac{a\left(1+\eta_1^{-z}\right)\phi\left(-z\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}} \equiv M$$

Define $\widehat{\eta} \equiv \eta_3 = \lim_{\mu_f\to1}\eta_2$. Then:

$$\begin{aligned}\lim_{\mu_f\to1} M &= \frac{a\left(1+\tau^*\right)^a\Phi^*}{\sigma-1}\left[\frac{2\widehat{\eta}^{-z}\left[\frac{2a\widehat{\eta}^{-a}}{(\sigma-1)(\kappa_f+\alpha_f)}\right] - 2\widehat{\eta}^{-a}\left[\frac{2z\widehat{\eta}^{-z}}{(\sigma-1)(\kappa_f+\alpha_f)}\right]}{4\widehat{\eta}^{-a-z}}\right] - \frac{2z\widehat{\eta}^{-z}}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}}\right]\\
&= \frac{1}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\left\{a\left(1+\tau^*\right)^a\Phi^* - 2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}}\right]\right\}\\
&= \frac{1}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\left\{a\left(1+\tau^*\right)^a\left(1+\eta_1^{-z}\right) + 2a\widehat{\eta}^{-z} - 2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{\left(1+\tau^*\right)\Gamma_{\tau^*}}\right]\right\}\end{aligned}$$

In equilibrium:

$$
\begin{aligned}
\Gamma\left(\tau^*\right)=0 \quad \Leftrightarrow \quad & 1+\eta_1^{-z}=\frac{\eta_2^{-z}+\eta_3^{-z}}{\left(1+\tau^*\right)^a\left(\tau^* a-\tau^*-1\right)} \\
\Rightarrow \quad & \lim_{\mu_f \rightarrow 1}\left(1+\tau^*\right)^a\left(1+\eta_1^{-z}\right)=\lim_{\mu_f \rightarrow 1}\left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\tau^* a-\tau^*-1}\right)=\frac{2 \widehat{\eta}^{-z}}{\tau^* a-\tau^*-1}
\end{aligned}
$$

Using this substitution means that:

$$
\lim_{\mu_f \rightarrow 1} M \quad=\quad \frac{2 \widehat{\eta}^{-z}}{(\sigma-1)\left(\kappa_f+\alpha_f\right)}\left\{\frac{a(a-1) \tau^*}{\tau^* a-\tau^*-1}-z\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{\left(1+\tau^*\right) \Gamma_{\tau^*}}\right]\right\}
$$

And:

$$
\frac{a\left(1+\eta_1^{-z}\right)}{\left(1+\tau^*\right) \Gamma_{\tau^*}}=\frac{-a\left(1+\eta_1^{-z}\right)}{(a-1)\left(1+\tau^*\right)\left(1+\eta_1^{-z}\right)+\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\left(1+\tau^*\right)^a}}=\frac{-a}{(a+1)(a-1) \tau^*-1}
$$

So:

$$
\begin{aligned}
\lim_{\mu_f \rightarrow 1} M \quad = \quad & \frac{2 \widehat{\eta}^{-z}}{(\sigma-1)\left(\kappa_f+\alpha_f\right)}\left\{\frac{a(a-1) \tau^*}{\tau^* a-\tau^*-1}-z\left[1+\frac{a}{(a+1)(a-1) \tau^*-1}\right]\right\} \\
= \quad & \frac{2 \widehat{\eta}^{-z}}{(\sigma-1)\left(\kappa_f+\alpha_f\right)}\left\{\frac{a(a-1) \tau^*}{(a-1) \tau^*-1}-z\left[\frac{(a+1)(a-1) \tau^*+a-1}{(a+1)(a-1) \tau^*-1}\right]\right\} \\
= \quad & \frac{2(a-1) \widehat{\eta}^{-z}}{(\sigma-1)\left(\kappa_f+\alpha_f\right)}\left\{a\left[\frac{\tau^*}{(a-1) \tau^*-1}\right]-z\left[\frac{(a+1) \tau^*+1}{(a+1)(a-1) \tau^*-1}\right]\right\} \\
& \frac{2(a-1) \widehat{\eta}^{-z}}{(\sigma-1)\left[(a-1) \tau^*-1\right]\left[(a+1)(a-1) \tau^*-1\right]\left(\kappa_f+\alpha_f\right)}\left\{a \tau^*\left[(a+1)(a-1) \tau^*-1\right]-z\left[(a+1) \tau^*+1\right]\left[(a-\right.\right. \\
& \frac{2(a-1) \widehat{\eta}^{-z}}{(\sigma-1)\left[(a-1) \tau^*-1\right]\left[(a+1)(a-1) \tau^*-1\right]\left(\kappa_f+\alpha_f\right)}\left\{a\left[\left(a^2-1\right)\left(\tau^*\right)^2-\tau^*\right]-z\left[\left(a^2-1\right)\left(\tau^*\right)^2-2 \tau^*-\right.\right. \\
& \frac{2(a-1) \widehat{\eta}^{-z}}{(\sigma-1)\left[(a-1) \tau^*-1\right]\left[(a+1)(a-1) \tau^*-1\right]\left(\kappa_f+\alpha_f\right)}\left\{(a-z)\left[\left(a^2-1\right) \tau^*-1\right] \tau^*+z\left(\tau^*+1\right)\right\}>0
\end{aligned}
$$

QED.

Proposition 6. For foreign firms, higher startup costs and higher mobility are associated with higher revenues.

Proof of Proposition 6

In equilibrium, the revenue of a foreign firm is:

$$
r_f^*(\varphi)=\left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} Q P^\sigma=\left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} L P\left(x_d\right)^{\sigma-1}=\frac{z L x_d^z \varphi^{\sigma-1}}{a b^a(1+\tau)^{\sigma-1} \Phi}
$$

The average revenue of an old foreign firm is:

$$\begin{aligned}
\overline{r}_f^{old} &= \frac{zLx_d^z}{ab^a\left(1+\tau\right)^{\sigma-1}\Phi}\int\limits_{x_f}^{\infty}\varphi^{\sigma-1}h_f^{old}d\varphi \quad \text{where: } h_f^{old}=\frac{g\left(\varphi\right)}{1-G\left(x_f\right)} \\
&= \frac{zLx_d^z}{ab^a\left(1+\tau\right)^{\sigma-1}\Phi\left[1-G\left(x_f\right)\right]}\left(\frac{ab^a}{z}x_f^{-z}\right) \\
&= \frac{Lx_d^z x_f^{\sigma-1}}{\left(1+\tau\right)^{\sigma-1}\Phi b^a} = \frac{L\eta_2^{\sigma-1}x_d^a}{b^a\Phi}
\end{aligned}$$

The average revenue of a new foreign firm is:

$$\begin{aligned}
\overline{r}_f^{new} &= \frac{zLx_d^z}{ab^a\left(1+\tau\right)^{\sigma-1}\Phi}\int\limits_{y_f}^{\infty}\varphi^{\sigma-1}h_f^{new}d\varphi \quad \text{where: } h_f^{new}=\frac{g\left(\varphi\right)}{1-G\left(y_f\right)} \\
&= \frac{zLx_d^z}{ab^a\left(1+\tau\right)^{\sigma-1}\Phi\left[1-G\left(y_f\right)\right]}\left(\frac{ab^a}{z}y_f^{-z}\right) \\
&= \frac{Lx_d^z y_f^{\sigma-1}}{b^a\left(1+\tau\right)^{\sigma-1}\Phi} = \frac{L\eta_3^{\sigma-1}x_d^a}{b^a\Phi}
\end{aligned}$$

So the average revenue of a foreign firm in the market in a given period is:

$$\begin{aligned}
\overline{r}_f &= \frac{M_f^{in}\overline{r}_f^{old}}{M_f}+\frac{M_f^{out}\overline{r}_f^{new}}{M_f} = \frac{1}{M_f}\left\{\left[1-G\left(x_f\right)\right]\left(\frac{L\eta_2^{\sigma-1}x_d^a}{b^a\Phi}\right)+\left[1-G\left(y_f\right)\right]\left(\frac{L\eta_3^{\sigma-1}x_d^a}{b^a\Phi}\right)\right\} \\
&= \frac{\left(1+\tau\right)^a x_d^a}{b^a\left(\eta_2^{-a}+\eta_3^{-a}\right)}\left(\frac{Lx_d^a}{b^a\Phi}\right)\left(b^a x_f^{-a}\eta_2^{\sigma-1}+b^a y_f^{-a}\eta_3^{\sigma-1}\right) = \frac{L\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d^a}{b^a\left(\eta_2^{-a}+\eta_3^{-a}\right)\Phi}
\end{aligned}$$

Foreign startup costs

The total effect of foreign start-up costs on foreign revenue will be:

$$\frac{dr_f\left(\varphi\right)}{d\kappa_f} = \frac{\partial r_f\left(\varphi\right)}{\partial\kappa_f}_{(+)} + \frac{\partial r_f\left(\varphi\right)}{\partial\tau^*}_{(-)}\left(\frac{\partial\tau^*}{\partial\kappa_f}\right)_{(-)}$$

where:

$$\begin{aligned}
\frac{\partial r_f\left(\varphi\right)}{\partial\tau^*} &= \frac{zL\varphi^{\sigma-1}}{ab^a}\left[\frac{\left(1+\tau\right)^{\sigma-1}\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau}-x_d^z\left[\left(1+\tau\right)^{\sigma-1}\frac{\partial\Phi}{\partial\tau}+\left(\sigma-1\right)\left(1+\tau\right)^{\sigma-2}\Phi\right]}{\left(1+\tau\right)^{2(\sigma-1)}\Phi^2}\right] \\
&= \frac{-zL\varphi^{\sigma-1}x_d^z}{ab^a}\left[\frac{z\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-a\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)+\left(\sigma-1\right)\Phi}{\left(1+\tau\right)^{\sigma}\Phi^2}\right] \\
&= \frac{-z\left(\sigma-1\right)L\varphi^{\sigma-1}x_d^z}{ab^a}\left[\frac{\Phi-\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\left(1+\tau\right)^{\sigma}\Phi^2}\right] \\
&= \frac{-z\left(\sigma-1\right)L\left(1+\eta_1^{-z}\right)x_d^z\varphi^{\sigma-1}}{ab^a\left(1+\tau\right)^{\sigma}\Phi^2} < 0
\end{aligned}$$

And:

$$\frac{\partial \tau^*}{\partial \kappa_f} = \frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}} = \frac{-z\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f} + \frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)}{(\sigma-1)(1+\tau)^a\left[(a-1)\left(1+\eta_1^{-z}\right) + a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]} < 0$$

And:

$$\frac{\partial r_f(\varphi)}{\partial \kappa_f} = \frac{zL\varphi^{\sigma-1}}{ab^a(1+\tau)^{\sigma-1}}\left[\frac{\Phi z x_d^{z-1}\frac{\partial x_d}{\partial \kappa_f} - x_d^z\frac{\partial \Phi}{\partial \kappa_f}}{\Phi^2}\right] = \frac{-zL\varphi^{\sigma-1}x_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi^2}\left(\frac{\sigma-1}{a}\right)\frac{\partial \Phi}{\partial \kappa_f} > 0$$

So: $\frac{dr_f(\varphi)}{d\kappa_f} > 0$.

Foreign mobility

The total effect of foreign mobility on foreign revenue will be:

$$\frac{dr_f(\varphi)}{d\mu_f} = \frac{\partial r_f(\varphi)}{\partial \mu_f}_{(+)} + \frac{\partial r_f(\varphi)}{\partial \tau^*}_{(-)}\left(\frac{\partial \tau^*}{\partial \mu_f}\right)_{(-)}$$

And:

$$\frac{\partial r_f(\varphi)}{\partial \mu_f} = \frac{zL\varphi^{\sigma-1}}{ab^a(1+\tau)^{\sigma-1}}\left[\frac{\Phi z x_d^{z-1}\frac{\partial x_d}{\partial \mu_f} - x_d^z\frac{\partial \Phi}{\partial \mu_f}}{\Phi^2}\right] = \frac{-zL\varphi^{\sigma-1}x_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi^2}\left(\frac{\sigma-1}{a}\right)\frac{\partial \Phi}{\partial \mu_f} > 0$$

So: $\frac{dr_f(\varphi)}{d\mu_f} > 0$.

QED.