"Firm Start-up Costs and Political Risks"

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Online Appendix

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Proposition 1. (Existence of a Market Equilibrium)
For any given takings rate in an industry j, $\tau_j \geq 0$, there exist types x_{ji} and y_{ji}, for each industry j and $i \in \{d, f\}$, such that $0 < x_{ji} < y_{ji}$. Firms that are in the market decide to exit if $\varphi < x_{ji}$, and stay and produce if $x_{ji} \leq \varphi$. Firms that are out of the market decide to stay out if $\varphi < y_{ji}$, and enter and produce if $y_{ji} \leq \varphi$.

Proof of Proposition 1.

1.1 Consumption

Recall that:

$$U = \left(1 - \sum_{j=1}^{J} w_j\right) \log v_0 + \sum_{j=1}^{J} w_j \log Q_j \quad \text{where: } Q_j \equiv \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}}$$

If the price of a variety in industry j is $p_j(v)$ and the quantity consumed of a variety in industry j is $q_j(v)$, then the budget constraint is:

$$\sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv \leq R$$

where R is the total revenue.

The Lagrangian is:

$$\mathcal{L} = \left(1 - \sum_{j=1}^{J} w_j\right) \log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda \left[R - \sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv\right]$$

Then constrained optimization for industry $j \in \{1, \ldots, J\}$ and product v' yields:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_j(v')} &= \frac{w_j}{Q_j} \left(\frac{\sigma}{\sigma-1}\right) \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}-1} \left(\frac{\sigma-1}{\sigma}\right) q_j(v')^{\frac{\sigma-1}{\sigma}-1} - \lambda p_j(v') \\
&= w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{-1} q_j(v')^{-\frac{1}{\sigma}} - \lambda p_j(v')
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \lambda} &= R - \sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv \\
\lambda &\geq 0 \\
\lambda \frac{\partial \mathcal{L}}{\partial \lambda} &= \lambda \left[R - \sum_{j=0}^{J} \int\limits_{v \in V_j} p_j(v)\, q_j(v)\, dv \right] = 0
\end{aligned}$$

Note that if $\lambda = 0$, then $\frac{\partial \mathcal{L}}{\partial q_j(v)} > 0$ for all levels of consumption. So it must be true that $\lambda > 0$ and $R - \sum_{j=0}^{J} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv = 0$.

Additionally, note that for any industry j and goods v, $v' \in V_j$, we have:

$$\begin{aligned}
\lambda &= w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v)^{-\frac{1}{\sigma}} p_j(v)^{-1} = w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} \\
&\Leftrightarrow \; q_j(v)^{-\frac{1}{\sigma}} = q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} p_j(v) \\
&\Leftrightarrow \; q_j(v)^{\frac{\sigma-1}{\sigma}} = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} p_j(v)^{1-\sigma} \\
&\Leftrightarrow \; \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} \int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \\
&\Leftrightarrow \; q_j(v')^{\frac{\sigma-1}{\sigma}} = p_j(v')^{1-\sigma} \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{-1} \\
&\Leftrightarrow \; q_j(v') = p_j(v')^{-\sigma} \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{-\frac{\sigma}{\sigma-1}} \\
&\Leftrightarrow \; q_j(v') = p_j(v')^{-\sigma} Q_j P_j^{\sigma} \quad \text{where: } P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}
\end{aligned}$$

So the demand function for any variety v in any industry j is:

$$q_j(v) = p_j(v)^{-\sigma} Q_j P_j^{\sigma}$$

where:

$$Q_j \equiv \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}$$

1.2 Production

Let the wage be 1, which represents one unit of the numeraire good, which is the unique good produced in industry $j = 0$. We suppress industry notation.

To make an output of q units, a domestic firm must use labor, $\frac{q}{\varphi}$, where φ is the firm's productivity. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by domestic firms is:

$$\pi_d(\varphi) = p_d(\varphi)\, q_d(\varphi) - \left[\frac{q_d(\varphi)}{\varphi} + c\right] = p_d(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_d(\varphi)^{-\sigma} QP^{\sigma}}{\varphi} + c\right]$$

Profit maximization by domestic firms yields:

$$\frac{\partial \pi_d(\varphi)}{\partial p_d} = (1-\sigma)\, p_d(\varphi)^{-\sigma} QP^{\sigma} + \sigma \left[\frac{p_d(\varphi)^{-\sigma-1} QP^{\sigma}}{\varphi}\right] = 0$$

$$\Leftrightarrow p_d(\varphi) = \frac{\rho}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma-1}$$

This yields revenue:

$$r_d(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_d(\varphi) &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma} - \left[\frac{\left(\frac{\varphi}{\rho}\right)^{\sigma} QP^{\sigma}}{\varphi} + c\right] \\ &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma} \left(1 - \frac{1}{\rho}\right) - c = \frac{r_d(\varphi)}{\sigma} - c\end{aligned}$$

To make an output of q units, a foreign firm must use labor, $\frac{q(1+\tau)}{\varphi}$, where φ is the firm's productivity and τ is the government taking rate. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by foreign firms is:

$$\pi_f(\varphi) = p_f(\varphi)\, q_f(\varphi) - \left[\frac{q_f(\varphi)(1+\tau)}{\varphi} + c\right] = p_f(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_f(\varphi)^{-\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right]$$

Profit maximization by foreign firms yields:

$$\frac{\partial \pi_f(\varphi)}{\partial p_f} = (1-\sigma)\, p_f(\varphi)^{-\sigma} QP^{\sigma} + \sigma \left[\frac{p_f(\varphi)^{-\sigma-1} QP^{\sigma}(1+\tau)}{\varphi}\right] = 0$$

$$\Leftrightarrow p_f(\varphi) = \frac{\rho(1+\tau)}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma-1}$$

This yields revenue:

$$r_f(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_f(\varphi) &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma} - \left[\frac{\left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right] \\ &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma} \left(1 - \frac{1}{\rho}\right) - c = \frac{r_f(\varphi)}{\sigma} - c\end{aligned}$$

So conditional on being in the market, optimal production by domestic firms yields:

$$p_d^*(\varphi) = \frac{\rho}{\varphi} \quad \text{and} \quad q_d^*(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma} QP^{\sigma} \quad \text{and} \quad r_d^*(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}$$

This yields domestic firm profit:

$$\pi_d^*(\varphi) = \phi_d \varphi^{\sigma-1} - c = \frac{r_d^*(\varphi)}{\sigma} - c \qquad \text{where: } \phi_d \equiv \frac{QP^{\sigma}}{\sigma \rho^{\sigma-1}}$$

And optimal production by foreign firms yields

$$p_f^*(\varphi) = \frac{\rho(1+\tau)}{\varphi} \quad \text{and} \quad q_f^*(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} QP^{\sigma} \quad \text{and} \quad r_f^*(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}$$

This yields foreign firm profit:

$$\pi_f^*(\varphi) = \phi_f \varphi^{\sigma-1} - c = \frac{r_f^*(\varphi)}{\sigma} - c \qquad \text{where: } \phi_f \equiv \frac{QP^{\sigma}}{\sigma \rho^{\sigma-1}(1+\tau)^{\sigma-1}}$$

1.3 Firm Entry and Exit

We suppress industry notation. First, note the following properties of the profit function:

$$\lim_{\varphi \to 0} \pi_i(\varphi) = -c < 0 \quad \text{and} \quad \lim_{\varphi \to \infty} \pi_i(\varphi) = \infty > 0$$

$$\text{and} \quad \frac{\partial \pi_i(\varphi)}{\partial \varphi} = (\sigma - 1)\phi_i \varphi^{\sigma-2} > 0$$

So function $\pi_i^*(\varphi)$ is invertible: for any $y \in \mathbf{R}$, there exists a unique $\varphi_y \in (0, \infty)$ such that $\pi_i^*(\varphi_y) = y$.
Define: $\Psi_i \equiv V_i^{in} - V_i^{out}$.
A firm that is already in the market has incentive to stay in the market (rather than exit) iff:

$$\mu_i \kappa_i + \delta V_i^{out} \leq \pi_i^*(\varphi) + \delta V_i^{in} \quad \Leftrightarrow \quad \mu_i \kappa_i - \delta \Psi_i \leq \pi_i^*(\varphi)$$

Define: $x_i \equiv \pi_i^{*-1}(\mu_i \kappa_i - \delta \Psi_i)$. This is the type of firm that is in the market and indifferent about whether to remain.
A firm that is not already in the market has incentive to enter the market iff:

$$\delta V_i^{out} \leq \pi_i^*(\varphi) - \kappa_i + \delta V_i^{in} \quad \Leftrightarrow \quad \kappa_i - \delta \Psi_i \leq \pi_i^*(\varphi)$$

Define: $y_i \equiv \pi_i^{*-1}(\kappa_i - \delta \Psi_i)$. This is the type of firm that is out of the market and indifferent about whether to enter.
Because $\pi_i^*(\varphi)$ is an increasing function and $\mu_i \kappa_i - \delta \Psi_i < \kappa_i - \delta \Psi_i$, it must be true that: $x_i < y_i$.

1.4 Weighted Average Productivity

We suppress industry notation. Define the weighted average productivity of firms as:

$$\widetilde{\varphi}_x = \widetilde{\varphi}(x) \equiv \left[\frac{1}{1-G(x)} \int_x^{\infty} \varphi^{\sigma-1} dG(\varphi)\right]^{\frac{1}{\sigma-1}} \quad \Leftrightarrow \quad \widetilde{\varphi}_x^{\sigma-1}[1-G(x)] = \int_x^{\infty} \varphi^{\sigma-1} dG(\varphi)$$

1.5 Continuation Values

We suppress industry notation. Define:

$$\alpha \equiv \frac{1}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}$$
$$\gamma_1 \equiv G(x_i)\left\{\mu_i\left[1-\delta G(y_i)\right]-\delta\left[1-G(y_i)\right]\right\}$$
$$\gamma_2 \equiv \left\{\delta\mu_i G(x_i)-\left[1-\delta+\delta G(x_i)\right]\right\}\left[1-G(y_i)\right]$$

The continuation values are:

$$V_i^{in} = \alpha\left\{\gamma_1\kappa_i+\left[1-G(x_i)\right](1-\delta G_y)\,\pi^*(\widetilde{\varphi}_x)+\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_y)\right\}$$

$$V_i^{out} = \alpha\left[\gamma_2\kappa_i+\delta\left[1-G(x_i)\right]\left[1-G(y_i)\right]\pi^*(\widetilde{\varphi}_x)+\left[1-\delta+\delta G(x_i)\right]\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_y)\right]$$

The details:

$$\begin{aligned}
V_i^{in} &= \int_b^{x_i}\left(\mu_i\kappa_i+\delta V_i^{out}\right)dG(\varphi)+\int_{x_i}^{\infty}\left[\pi_i^*(\varphi)+\delta V_i^{in}\right]dG(\varphi)\\
&= \left(\mu_i\kappa_i+\delta V_i^{out}\right)G(x_i)+\delta V_i^{in}\left[1-G(x_i)\right]+\int_{x_i}^{\infty}\pi_i^*(\varphi)\,dG(\varphi)\\
&= \frac{1}{1-\delta\left[1-G(x_i)\right]}\left\{\left(\mu_i\kappa_i+\delta V_i^{out}\right)G(x_i)+\pi^*(\widetilde{\varphi}_{xi})\left[1-G(x_i)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
V_i^{out} &= \int_0^{y_i}\delta V_i^{out}dG(\varphi)+\int_{y_i}^{\infty}\left[\pi_i^*(\varphi)-\kappa_i+\delta V_i^{in}\right]dG(\varphi)\\
&= \delta V_i^{out}G(y_i)+\left(\delta V_i^{in}-\kappa_i\right)\left[1-G(y_i)\right]+\int_{y_i}^{\infty}\pi_i^*(\varphi)\,dG(\varphi)\\
&= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

Substitution:

$$\begin{aligned}
\left\{1-\delta\left[1-G(x_i)\right]\right\}V_i^{in} &= G(x_i)\,\mu_i\kappa_i+\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left(\frac{1-G(y_i)}{1-\delta G(y_i)}\right)\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

$$\begin{aligned}
\left\{1-\delta\left[1-G(x_i)\right]\right\}\left[1-\delta G(y_i)\right]V_i^{in} &= G(x_i)\left[1-\delta G(y_i)\right]\mu_i\kappa_i+\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left[1-G(y_i)\right]\left[\delta V_i^{in}-\kappa_i+\pi_i^*(\widetilde{\varphi}_{yi})\right]
\end{aligned}$$

$$\begin{aligned}
&V_i^{in}\left\langle\left\{1-\delta\left[1-G(x_i)\right]\right\}\left[1-\delta G(y_i)\right]-\delta^2 G(x_i)\left[1-G(y_i)\right]\right\rangle\\
&\quad=\left\{G(x_i)\left[1-\delta G(y_i)\right]\mu_i-\delta G(x_i)(1-G_{yi})\right\}\kappa_i+\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi})\\
&\quad+\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})
\end{aligned}$$

$$
\begin{aligned}
&V_i^{in}(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right] \\
&\quad = \left\{\left[1-\delta G(y_i)\right]\mu_i - \delta\left[1-G(y_i)\right]\right\} G(x_i)\kappa_i + \left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi}) \\
&\quad +\delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})
\end{aligned}
$$

$$
\begin{aligned}
V_i^{in} \quad = \quad & \frac{1}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\langle \left\{\left[1-\delta G(y_i)\right]\mu_i - \delta\left[1-G(y_i)\right]\right\} G(x_i)\kappa_i \right. \\
& \left. + \left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi}) + \delta G(x_i)\left[1-G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{yi})\right\rangle
\end{aligned}
$$

And:

$$
\begin{aligned}
V_i^{out} &= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\frac{\delta\left\{\left[1-\delta G(y_i)\right]\mu_i - \delta\left[1-G(y_i)\right]\right\} G(x_i)\kappa_i + \delta\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi}) + \delta^2 G(x_i)\left[1-G(\right.}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\right. \\
&= \frac{1-G(y_i)}{1-\delta G(y_i)}\left[\frac{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\left[1-\delta G(y_i)\right]\kappa_i + \delta\left[1-G(x_i)\right]\left[1-\delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{xi}) + \left[1-\delta+\delta G(x_i)\right]\left[\right.}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\right. \\
&= \frac{1-G(y_i)}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\kappa_i + \delta\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi}) + \left[1-\delta+\delta G(x_i)\right]\pi_i^*(\widetilde{\varphi}_{yi}\right.
\end{aligned}
$$

$$
V_i^{out} = \frac{1-G(y_i)}{(1-\delta)\left[1+\delta G(x_i)-\delta G(y_i)\right]}\left\{-\left[1-\delta+\delta G(x_i)(1-\mu_i)\right]\kappa_i + \delta\left[1-G(x_i)\right]\pi^*(\widetilde{\varphi}_{xi}) + \left[1-\delta+\delta G(x_i)\right]\pi_i^*(\widetilde{\varphi}_{yi}\right.
$$

1.6 Free Entry

We suppress industry notation. Free entry (FE) requires that $V_i^{in} = \beta_i^{in}$ and $V_i^{out} = \beta_i^{out}$. So it must be true that $\Psi_i(x,y) = \beta_i^{in} - \beta_i^{out} = \Psi_i$ for $i \in \{d,f\}$.

1.7 Zero Profit Conditions

We suppress industry notation. Because "in" firms are indifferent at cutpoint x_i, and "out" firms are indifferent at cutpoint y_i, we have four zero profit conditions (ZPCs):

$$
\begin{aligned}
\pi_i^*(x_i) &= \mu_i\kappa_i - \delta\Psi_i \\
\pi_i^*(y_i) &= \kappa_i - \delta\Psi_i \qquad \text{for: } i \in \{d,f\}
\end{aligned}
$$

These imply that:

$$
\begin{aligned}
r_i^*(x_i) &= \sigma(\mu_i\kappa_i + \alpha_i) \\
r_i^*(y_i) &= \sigma(\kappa_i + \alpha_i) \qquad \text{for: } i \in \{d,f\} \text{ and } \alpha_i \equiv c - \delta\Psi_i > 0
\end{aligned}
$$

Note that:

$$
\frac{r_d^*(x_d)}{r_d^*(y_d)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left(\frac{y_d}{\rho}\right)^{\sigma-1} QP^\sigma} = \left(\frac{x_d}{y_d}\right)^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\kappa_d + \alpha_d)} \quad \Leftrightarrow \quad y_d = x_d\left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}
$$

Note that: $x_d < y_d \Leftrightarrow \mu_d < 1$, which always holds.

$$
\frac{r_d^*(x_d)}{r_f^*(x_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left[\frac{x_f}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma} = \left[\frac{x_d(1+\tau)}{x_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\mu_f\kappa_f + \alpha_f)} \quad \Leftrightarrow \quad x_f = x_d(1+\tau)\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}
$$

Note that: $x_d < x_f \Leftrightarrow \mu_d\kappa_d - \delta\Psi_d + c < (1+\tau_f)^{\sigma-1}(\mu_f\kappa_f - \delta\Psi_f + c)$

$$\frac{r_d^*(x_d)}{r_f^*(y_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left[\frac{y_f}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma} = \left[\frac{x_d(1+\tau)}{y_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\kappa_f + \alpha_f)} \quad \Leftrightarrow \quad y_f = x_d(1+\tau)\left(\frac{\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}$$

Note that $x_f < y_f \Leftrightarrow \mu_f < 1$.

So the cutpoints (y_d, x_f, y_f) can all be expressed in terms of variable x_d:

$$\begin{aligned}
y_d(x_d) &= \eta_1 x_d \qquad \text{where: } \eta_1 \equiv \left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\
x_f(x_d) &= \eta_2(1+\tau)x_d \qquad \text{where: } \eta_2 \equiv \left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\
y_f(x_d) &= \eta_3(1+\tau)x_d \qquad \text{where: } \eta_3 \equiv \left(\frac{\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}
\end{aligned}$$

Recall that we have suppressed the industry subscript. So each industry has a different set of cutpoints.

1.8 Firm Masses

We suppress industry notation. The total mass of i firms is:

$$M_i = M_i^{in} + M_i^{out}$$

Stationarity requires the following:

$$\begin{aligned}
M_i^{in} &= [1 - G(x_i)]M_i^{in} + [1 - G(y_i)]M_i^{out} \\
&\Leftrightarrow \quad M_i^{in} = \left(\frac{1 - G(y_i)}{G(x_i)}\right)M_i^{out} \\
&\Leftrightarrow \quad M_i^{in} = \left(\frac{1 - G(y_i)}{G(x_i)}\right)(M_i - M_i^{in}) \\
&\Leftrightarrow \quad M_i^{in}\left(\frac{G(x_i) + 1 - G(y_i)}{G(x_i)}\right) = \left(\frac{1 - G(y_i)}{G(x_i)}\right)M_i \\
&\Leftrightarrow \quad M_i^{in} = \left(\frac{1 - G(y_i)}{G(x_i) + 1 - G(y_i)}\right)M_i
\end{aligned}$$

$$\begin{aligned}
M_i^{out} &= G(x_i)M_i^{in} + G(y_i)M_i^{out} \\
&\Leftrightarrow \quad M_i^{out} = \left(\frac{G(x_i)}{1 - G(y_i)}\right)M_i^{in} = \left(\frac{G(x_i)}{1 - G(y_i)}\right)\left(\frac{1 - G(y_i)}{G(x_i) + 1 - G(y_i)}\right)M_i \\
&\Leftrightarrow \quad M_i^{out} = \left(\frac{G(x_i)}{G(x_i) + 1 - G(y_i)}\right)M_i
\end{aligned}$$

1.9 Labor Market Clearing

We suppress industry notation. Note that the distribution of new producers (firms that were "out", but then entered) differs from the distribution of old producers (firms that were "in" and then stayed). Namely:

$$h_i^{old} = \begin{cases} \frac{g(\varphi)}{1-G(x_i)} & \text{if } x_i \le \varphi \\ 0 & \text{if } \varphi < x_i \end{cases} \quad \text{and} \quad h_i^{new} = \begin{cases} \frac{g(\varphi)}{1-G(y_i)} & \text{if } y_i \le \varphi \\ 0 & \text{if } \varphi < y_i \end{cases}$$

Recall that:

$$\widetilde{\varphi}_x = \widetilde{\varphi}(x) \equiv \left[\frac{1}{1-G(x)} \int_x^\infty \varphi^{\sigma-1} dG(\varphi) \right]^{\frac{1}{\sigma-1}} \quad \Leftrightarrow \quad \widetilde{\varphi}_x^{\sigma-1} [1-G(x)] = \int_x^\infty \varphi^{\sigma-1} dG(\varphi)$$

Aggregate profit for i firms in a specific period t is:

$$\begin{aligned}
\Pi_i^t &= \underbrace{[1-G(x_i)] M_i^{in}}_{\text{mass of old firms (that stay)}} \underbrace{\int_{x_i}^\infty \pi_i^*(\varphi) h_i^{old}(\varphi) d\varphi}_{\text{old firm profits}} + \underbrace{[1-G(y_i)] M_i^{out}}_{\text{mass of new firms}} \underbrace{\int_{y_i}^\infty \pi_i^*(\varphi) h_i^{new}(\varphi) d\varphi}_{\text{new firm profits}} \\
&= M_i^{in} \int_{x_i}^\infty \left(\phi_i \varphi^{\sigma-1} - c\right) dG(\varphi) + M_i^{out} \int_{y_i}^\infty \left(\phi_i \varphi^{\sigma-1} - c\right) dG(\varphi) \\
&= \phi_i M_i^{in} \int_{x_i}^\infty \varphi^{\sigma-1} g(\varphi) d\varphi - c M_i^{in} [1-G(x_i)] + \phi_i M_i^{out} \int_{y_i}^\infty \varphi^{\sigma-1} g(\varphi) d\varphi - c M_i^{out} [1-G(y_i)] \\
&= \phi_i M_i^{in} \widetilde{\varphi}_{xi}^{\sigma-1} [1-G(x_i)] - c M_i^{in} [1-G(x_i)] + \phi_i M_i^{out} \widetilde{\varphi}_{yi}^{\sigma-1} [1-G(y_i)] - c M_i^{out} [1-G(y_i)] \\
&= M_i^{in} \left(\phi_i \widetilde{\varphi}_{xi}^{\sigma-1} - c\right) [1-G(x_i)] + M_i^{out} \left(\phi_i \widetilde{\varphi}_{yi}^{\sigma-1} - c\right) [1-G(y_i)] \\
&= M_i^{in} \pi_i^*(\widetilde{\varphi}_{xi}) [1-G(x_i)] + M_i^{out} \pi_i^*(\widetilde{\varphi}_{yi}) [1-G(y_i)]
\end{aligned}$$

The present value of aggregate profit for i firms over time is therefore:

$$\Pi_i = M_i^{in} \pi_i^*(\widetilde{\varphi}_{xi}) [1-G(x_i)] + M_i^{out} \pi_i^*(\widetilde{\varphi}_{yi}) [1-G(y_i)] + \delta M_i^{in} V_i^{in} + \delta M_i^{out} V_i^{out}$$

We can now invoke the free entry conditions:

$$\beta_i^{in} = V_i^{in} = \alpha \left\{ \gamma_1 \kappa_i + [1-G(x_i)] [1-\delta G(y_i)] \pi^*(\widetilde{\varphi}_{xi}) + \delta G(x_i) [1-G(y_i)] \pi_i^*(\widetilde{\varphi}_{yi}) \right\}$$

$$\beta_i^{out} = V_i^{out} = \alpha \left\{ \gamma_2 \kappa_i + \delta [1-G(x_i)] [1-G(y_i)] \pi^*(\widetilde{\varphi}_{xi}) + [1-\delta+\delta G(x_i)] [1-G(y_i)] \pi_i^*(\widetilde{\varphi}_{yi}) \right\}$$

By manipulating these conditions, we can isolate $\pi^*(\widetilde{\varphi}_{xi})$ and $\pi^*(\widetilde{\varphi}_{yi})$ to show that:

$$\pi^*(\widetilde{\varphi}_{xi}) [1-G(x_i)] = [1-\delta+\delta G(x_i)] \beta_i^{in} - \delta G(x_i) \beta_i^{out} - G(x_i) \mu_i \kappa_i \qquad (\star)$$

$$\pi_i^*(\widetilde{\varphi}_{yi}) [1-G(y_i)] = [1-\delta G(y_i)] \beta_i^{out} - \delta [1-G(y_i)] \beta_i^{in} + [1-G(y_i)] \kappa_i$$

Substitution yields:

$$\begin{aligned}
\Pi_i &= M_i^{in} \left\{ [1-\delta+\delta G(x_i)] \beta_i^{in} - \delta G(x_i) \beta_i^{out} - G(x_i) \mu_i \kappa_i \right\} \\
&\quad + M_i^{out} \left\{ [1-\delta G(y_i)] \beta_i^{out} - \delta [1-G(y_i)] \beta_i^{in} + [1-G(y_i)] \kappa_i \right\} + \delta M_i^{in} \beta_i^{in} + \delta M_i^{out} \beta_i^{out}
\end{aligned}$$

$$\begin{aligned}
\Pi_i &= \left\{ [1+\delta G(x_i)] M_i^{in} - \delta [1-G(y_i)] M_i^{out} \right\} \beta_i^{in} \\
&\quad + \left\{ [1+\delta-\delta G(y_i)] M_i^{out} - \delta G(x_i) M_i^{in} \right\} \beta_i^{out} + [1-G(y_i)] \kappa_i M_i^{out} - G(x_i) \mu_i \kappa_i M_i^{in}
\end{aligned}$$

$$\begin{aligned}\Pi_i &= \left\{[1+\delta G(x_i)]M_i^{in} - \delta[1-G(y_i)]M_i^{out}\right\}\beta_i^{in} + [1-G(y_i)]M_i^{out}\kappa_i - G_x M_i^{in}\mu_i\kappa_i \\ &\quad + \left\{[1+\delta-\delta G(y_i)]M_i^{out} - \delta G(x_i)M_i^{in}\right\}\beta_i^{out}\end{aligned}$$

Using the equilibrium masses yields:

$$\begin{aligned}\Pi_i &= \frac{M_i}{G(x_i)+1-G(y_i)}\left\{[1+\delta G(x_i)][1-G(y_i)] - \delta[1-G(y_i)]G(x_i)\right\}\beta_i^{in} + [1-G(y_i)]M_i^{out}\kappa_i - G(x_i)M_i^{in}\mu_i\kappa_i \\ &+ \frac{M_i}{G(x_i)+1-G(y_i)}\left\{[1+\delta-\delta G(y_i)]G(x_i) - \delta G(x_i)[1-G(y_i)]\right\}\beta_i^{out}\end{aligned}$$

$$\begin{aligned}\Pi_i &= \frac{M_i[1-G(y_i)]}{G(x_i)+1-G(y_i)}\beta_i^{in} + \frac{M_i G(x_i)}{G(x_i)+1-G(y_i)}\beta_i^{out} + [1-G(y_i)]M_i^{out}\kappa_i - G(x_i)M_i^{in}\mu_i\kappa_i \\ &= \underbrace{M_i^{in}\beta_i^{in} + M_i^{out}\beta_i^{out}}_{L_{ti}} + \underbrace{[1-G(y_i)]M_i^{out}\kappa_i}_{L_{si}} - \underbrace{G(x_i)M_i^{in}\mu_i\kappa_i}_{L_{ri}}\end{aligned}$$

where:

- L_{ti} is labor spent on learning each firm's type
- L_{si} is labor that "out" firms spend on setting up new production when they enter the market
- L_{ri} is labor that is recovered when "in" firms decide to exit the market

Total revenue by i firms is the total profits, plus the total production costs, L_{pi}. Then:

$$R_i = \Pi_i + L_{pi} = L_{pi} + L_{ti} + L_{si} - L_{ri} = L_i$$

So the labor market for a given industry clears.

Note that given the equilibrium demand function:

$$\begin{aligned}\int\limits_{v\in V_j} p_j(v)\,q_j(v)\,dv &= \int\limits_{v\in V_j} p_j(v)^{1-\sigma} Q_j P_j^{\sigma}\,dv \\ &= Q_j P_j^{\sigma}\int\limits_{v\in V_j} p_j(v)^{1-\sigma}\,dv \\ &= Q_j P_j^{\sigma} P_j^{1-\sigma} = P_j Q_j\end{aligned}$$

So by going back to our original Lagrangian, we can see that:

$$\begin{aligned}\mathcal{L} &= \left(1-\sum_{j=1}^{J} w_j\right)\log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda\left[R - \sum_{j=0}^{J}\int\limits_{v\in V_j} p_j(v)\,q_j(v)\,dv\right] \\ &= \left(1-\sum_{j=1}^{J} w_j\right)\log v_0 + \sum_{j=1}^{J} w_j \log Q_j + \lambda\left[R - \sum_{j=0}^{J} P_j Q_j\right]\end{aligned}$$

Then for $j \in \{1,\ldots,J\}$, the FOC becomes:

$$\frac{\partial \mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \lambda P_j = 0 \quad \Leftrightarrow \quad \lambda = \frac{w_j}{P_j Q_j}$$

And:

$$1 = \sum_{j=0}^{J} w_j = \lambda \sum_{j=0}^{J} P_j Q_j = \lambda \sum_{j=0}^{J} R_j = \lambda R = \lambda L \quad \Leftrightarrow \quad \lambda = \frac{1}{L}$$

So:

$$\frac{\partial \mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \frac{P_j}{L} = 0 \quad \Leftrightarrow \quad L w_j = P_j Q_j = R_j = L_i$$

So the labor market across all industries clears.

1.10 Equilibrium Characterization under the Pareto Distribution

We suppress industry notation. Suppose that types are chosen according to the Pareto distribution, iid over time and players, with domain $\varphi \sim [b, \infty)$ for small $b > 0$.

$$\begin{aligned} \text{Density function:} \quad & g(\varphi) &=& \frac{ab^a}{\varphi^{a+1}} \quad \text{for: } \varphi \geq b \\ \text{Distribution function:} \quad & G(y) &=& \Pr(\varphi \leq y) = 1 - \left(\frac{b}{y}\right)^a \quad \text{for: } x \geq b \end{aligned}$$

Equilibrium behavior is defined by the function:

$$\pi_d^*(x_d) = \phi_d x_d^{\sigma-1} - c = \mu_d \kappa_d - \delta \Psi_d$$

where:

$$\phi_d = \frac{QP^\sigma}{\sigma \rho^{\sigma-1}} = \frac{RP^{\sigma-1}}{\sigma \rho^{\sigma-1}} = \frac{LP^{\sigma-1}}{\sigma \rho^{\sigma-1}}$$

Define $z \equiv a - \sigma + 1$. Under the Pareto distribution, for $x \geq b$:

$$\begin{aligned} \int_x^\infty \varphi^{\sigma-1} dG(\varphi) &= \int_x^\infty \varphi^{\sigma-1} \left(\frac{ab^a}{\varphi^{a+1}}\right) d\varphi = ab^a \int_x^\infty \varphi^{\sigma-a-2} d\varphi \\ &= ab^a \lim_{t\to\infty} \int_x^t \varphi^{\sigma-a-2} d\varphi = ab^a \lim_{t\to\infty} \left[\frac{\varphi^{\sigma-a-1}}{\sigma-a-1}\right]_x^t = -\frac{ab^a}{z} \lim_{t\to\infty} \left[\frac{1}{t^z} - \frac{1}{x^z}\right] \\ &= \frac{ab^a}{z}\left(\frac{1}{x^z}\right) = \frac{ab^a}{z} x^{-z} \end{aligned}$$

The equilibrium price index is:

$$\begin{aligned} P(x_d) &= \rho \left[\int_{x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + \int_{\eta_1 x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + (1+\tau)^{1-\sigma} \left(\int_{\eta_2(1+\tau)x_d}^\infty \varphi^{\sigma-1} dG(\varphi) + \int_{\eta_3(1+\tau)x_d}^\infty \varphi^{\sigma-1} dG(\varphi)\right)\right]^{\frac{1}{1-\sigma}} \\ &= \rho \left\{\frac{ab^a}{z} x_d^{-z} + \frac{ab^a}{z}(\eta_1 x_d)^{-z} + (1+\tau)^{1-\sigma}\left[\frac{ab^a}{z}(\eta_2(1+\tau)x_d)^{-z} + \frac{ab^a}{z}(\eta_3(1+\tau)x_d)^{-z}\right]\right\}^{\frac{1}{1-\sigma}} \\ &= \rho \left(\frac{z x_d^z}{ab^a \Phi}\right)^{\frac{1}{\sigma-1}} \qquad \text{where: } \Phi \equiv 1 + \eta_1^{-z} + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right) \end{aligned}$$

And:

$$P\left(x_d\right)^{\sigma-1} = \frac{\rho^{\sigma-1} z x_d^z}{ab^a \Phi}$$

So the best response function is:

$$\begin{aligned}
\Rightarrow \pi_d^*\left(x_d\right) = \frac{LP^{\sigma-1}}{\sigma\rho^{\sigma-1}} x_d^{\sigma-1} - c = \mu_d\kappa_d - \delta\Psi_d \\
\Leftrightarrow \frac{L}{\sigma\rho^{\sigma-1}}\left(\frac{\rho^{\sigma-1} z x_d^z}{ab^a\Phi}\right) x_d^{\sigma-1} = \mu_d\kappa_d + \alpha_d \\
\Leftrightarrow x_d^*\left(\tau\right) = \psi\Phi^{\frac{1}{a}} \quad \text{where: } \psi \equiv \left[\frac{\sigma ab^a}{Lz}\left(\mu_d\kappa_d + \alpha_d\right)\right]^{\frac{1}{a}}
\end{aligned}$$

QED.

Intermediate Calculations for Comparative Statics

Cutpoint scalars

$$\begin{aligned}
\eta_1 &= \left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_1}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\left(\mu_d\kappa_d + \alpha_d\right) - \mu_d\left(\kappa_d + \alpha_d\right)}{\left(\mu_d\kappa_d + \alpha_d\right)^2}\right) = \frac{\left(1-\mu_d\right)\alpha_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d + \alpha_d\right)\left(\kappa_d + \alpha_d\right)} \\
\frac{\partial\eta_1}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\kappa_d + \alpha_d\right)}{\left(\mu_d\kappa_d + \alpha_d\right)^2}\right) = \frac{-\kappa_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d + \alpha_d\right)}
\end{aligned}$$

$$\begin{aligned}
\eta_2 &= \left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_2}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\mu_d\left(\mu_f\kappa_f + \alpha_f\right)}{\left(\mu_d\kappa_d + \alpha_d\right)^2}\right) = \frac{-\mu_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d + \alpha_d\right)} \\
\frac{\partial\eta_2}{\partial\kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\mu_f}{\mu_d\kappa_d + \alpha_d}\right) = \frac{\mu_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f + \alpha_f\right)} \\
\frac{\partial\eta_2}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\mu_f\kappa_f + \alpha_f\right)}{\left(\mu_d\kappa_d + \alpha_d\right)^2}\right) = \frac{-\kappa_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d + \alpha_d\right)} \\
\frac{\partial\eta_2}{\partial\mu_f} &= \frac{1}{\sigma-1}\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\kappa_f}{\mu_d\kappa_d + \alpha_d}\right) = \frac{\kappa_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f + \alpha_f\right)}
\end{aligned}$$

$$
\begin{aligned}
\eta_3 &= \left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}} \\
\frac{\partial\eta_3}{\partial\kappa_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\mu_d\left(\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\mu_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)} \\
\frac{\partial\eta_3}{\partial\kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{1}{\mu_d\kappa_d+\alpha_d}\right)=\frac{\eta_3}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)} \\
\frac{\partial\eta_3}{\partial\mu_d} &= \frac{1}{\sigma-1}\left(\frac{\kappa_f+\alpha_f}{\mu_d\kappa_d+\alpha_d}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{-\kappa_d\left(\kappa_f+\alpha_f\right)}{\left(\mu_d\kappa_d+\alpha_d\right)^2}\right)=\frac{-\kappa_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}
\end{aligned}
$$

<u>Φ partials</u>

$$
\Phi\left(\tau\right)\equiv 1+\eta_1^{-z}+\left(1+\tau\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial\tau}=-a\left(1+\tau\right)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial y}=-z\left[\eta_1^{-z-1}\frac{\partial\eta_1}{\partial y}+\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\frac{\partial\eta_2}{\partial y}+\eta_3^{-z-1}\frac{\partial\eta_3}{\partial y}\right)\right]\text{ for: } y=\kappa_f,\kappa_d,\mu_f,\mu_d
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\kappa_f} &= -z\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\left[\frac{\mu_f\eta_2}{\left(\sigma-1\right)\left(\mu_f\kappa_f+\alpha_f\right)}\right]+\eta_3^{-z-1}\left[\frac{\eta_3}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\right]\right) \\
&= \frac{-z}{\left(\sigma-1\right)\left(1+\tau\right)^a}\left(\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\kappa_d} &= -z\left\{\eta_1^{-z-1}\left[\frac{\left(1-\mu_d\right)\alpha_d\eta_1}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)\left(\kappa_d+\alpha_d\right)}\right]+\left(1+\tau\right)^{-a}\left(\eta_2^{-z-1}\left[\frac{-\mu_d\eta_2}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\right]+\eta_3^{-z-1}\left[\frac{-\mu_d\eta_3}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\right.}\right.\right.\right. \\
&= \frac{-z}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\left[\frac{\left(1-\mu_d\right)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d}-\frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\left(1+\tau\right)^a}\right]
\end{aligned}
$$

$$
\frac{\partial\Phi\left(\tau\right)}{\partial\mu_f}=\frac{-z\kappa_f\eta_2^{-z}}{\left(\sigma-1\right)\left(1+\tau\right)^a\left(\mu_f\kappa_f+\alpha_f\right)}
$$

$$
\begin{aligned}
\frac{\partial\Phi\left(\tau\right)}{\partial\mu_d} &= -z\left[\frac{-\kappa_d\eta_1^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}+\left(1+\tau\right)^{-a}\left(\frac{-\kappa_d\eta_2^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}+\frac{-\kappa_d\eta_3^{-z}}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\right)\right] \\
&= \frac{z\kappa_d}{\left(\sigma-1\right)\left(\mu_d\kappa_d+\alpha_d\right)}\left[\eta_1^{-z}+\frac{\eta_2^{-z}+\eta_3^{-z}}{\left(1+\tau\right)^{-a}}\right]
\end{aligned}
$$

Cutpoint x_d

$$\frac{\partial x_d^*}{\partial \tau} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \tau} = \frac{-\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d}{(1+\tau)^{a+1}\Phi}$$

$$\frac{\partial x_d^*}{\partial \kappa_f} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \kappa_f} = \frac{x_d}{a\Phi}\left(\frac{\partial \Phi}{\partial \kappa_f}\right) = \frac{-zx_d}{a(\sigma-1)(1+\tau)^a\Phi}\left(\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)$$

$$\frac{\partial x_d^*}{\partial \mu_f} = \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \mu_f} = \frac{-z\kappa_f\eta_2^{-z}x_d}{a(\sigma-1)(1+\tau)^a(\mu_f\kappa_f+\alpha_f)\Phi}$$

$$\begin{aligned}
\frac{\partial x_d^*}{\partial \kappa_d} &= \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \kappa_d} + \frac{\partial \psi}{\partial \kappa_d}\Phi^{\frac{1}{a}} \\
&= \frac{-zx_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d}-\frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right]+\frac{1}{a}\left[\frac{\sigma ab^a}{Lz}(\mu_d\kappa_d+\alpha_d)\right]^{\frac{1}{a}-1}\left(\frac{\sigma ab^a\mu_d}{Lz}\right)\Phi^{\frac{1}{a}} \\
&= \frac{-zx_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d}-\frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right]+\frac{x_d}{a}\left(\frac{\mu_d}{\mu_d\kappa_d+\alpha_d}\right) \\
&= \frac{x_d}{a(\mu_d\kappa_d+\alpha_d)}\left\{\mu_d-\frac{z}{(\sigma-1)\Phi}\left[\frac{(1-\mu_d)\alpha_d\eta_1^{-z}}{\kappa_d+\alpha_d}-\frac{\mu_d\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^a}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_d^*}{\partial \mu_d} &= \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\frac{\partial \Phi}{\partial \mu_d} + \frac{\partial \psi}{\partial \mu_d}\Phi^{\frac{1}{a}} \\
&= \frac{z\kappa_d x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\eta_1^{-z}+\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right]+\frac{1}{a}\left[\frac{\sigma ab^a}{Lz}(\mu_d\kappa_d+\alpha_d)\right]^{\frac{1}{a}-1}\left(\frac{\sigma ab^a\kappa_d}{Lz}\right)\Phi^{\frac{1}{a}} \\
&= \frac{z\kappa_d x_d}{a(\sigma-1)(\mu_d\kappa_d+\alpha_d)\Phi}\left[\eta_1^{-z}+\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right]+\frac{x_d}{a}\left(\frac{\kappa_d}{\mu_d\kappa_d+\alpha_d}\right) \\
&= \frac{\kappa_d x_d}{a(\mu_d\kappa_d+\alpha_d)}\left\{1+\frac{z}{(\sigma-1)\Phi}\left[\eta_1^{-z}+\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau)^{-a}}\right]\right\}
\end{aligned}$$

Proposition 2. (Comparative Statics on the Takings Rate)
When the government increases the takings rate:

(a) firm-level foreign production and profits decrease, while firm-level domestic production and profits increase;

(b) aggregate foreign production decreases, while aggregate domestic production increases; and

(c) average observed productivity for foreign firms increases, while average observed productivity for domestic firms decreases.

Proof of Proposition 2.

(a) Firm-level production and profit

Firm-level foreign production is:

$$
\begin{aligned}
q_f(\varphi,\tau) &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} Q(x_d(\tau))\,P(x_d(\tau))^{\sigma} = \frac{L\varphi^{\sigma}P(x_d(\tau))^{\sigma-1}}{\rho^{\sigma}(1+\tau)^{\sigma}} = \frac{Lzx_d^z\varphi^{\sigma}}{ab^a\rho(1+\tau)^{\sigma}\Phi} \\
\frac{\partial q_f(\varphi,\tau)}{\partial\tau} &= \frac{Lz\varphi^{\sigma}}{ab^a\rho}\left\{\frac{(1+\tau)^{\sigma}\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau} - x_d^z\left[(1+\tau)^{\sigma}\frac{\partial\Phi}{\partial\tau}+\sigma(1+\tau)^{\sigma-1}\Phi\right]}{(1+\tau)^{2\sigma}\Phi^2}\right\} \\
&= \frac{Lz\varphi^{\sigma}x_d^z}{ab^a\rho}\left[\frac{a(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-\sigma\Phi-z(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{\sigma+1}\Phi^2}\right] \\
&= \frac{Lz\varphi^{\sigma}x_d^z}{ab^a\rho}\left[\frac{(\sigma-1)(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-\sigma\Phi}{(1+\tau)^{\sigma+1}\Phi^2}\right] \\
&= \frac{-Lz\varphi^{\sigma}x_d^z}{ab^a\rho}\left[\frac{\sigma\left(1+\eta_1^{-z}\right)+(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{\sigma+1}\Phi^2}\right] < 0
\end{aligned}
$$

Firm-level foreign profit production is:

$$
\begin{aligned}
\pi_f(\varphi,\tau) &= \left[\frac{Q(x_d)\,P(x_d)^{\sigma}}{\sigma\rho^{\sigma-1}(1+\tau)^{\sigma-1}}\right]\varphi^{\sigma-1} - c = \frac{L\varphi^{\sigma-1}P(x_d)^{\sigma-1}}{\sigma\rho^{\sigma-1}(1+\tau)^{\sigma-1}} - c = \frac{Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma(1+\tau)^{\sigma-1}\Phi} - c \\
\frac{\partial\pi_f(\varphi,\tau)}{\partial\tau} &= \frac{Lz\varphi^{\sigma-1}}{ab^a\sigma}\left\{\frac{(1+\tau)^{\sigma-1}\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau} - \left[(1+\tau)^{\sigma-1}\frac{\partial\Phi}{\partial\tau}+(\sigma-1)(1+\tau)^{\sigma-2}\Phi\right]x_d^z}{(1+\tau)^{2(\sigma-1)}\Phi^2}\right\} \\
&= \frac{Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma}\left[\frac{a(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-(\sigma-1)\Phi-z(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{\sigma}\Phi^2}\right] \\
&= \frac{(\sigma-1)Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma}\left[\frac{(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-\Phi}{(1+\tau)^{\sigma}\Phi^2}\right] == \frac{-(\sigma-1)Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma}\left[\frac{1+\eta_1^{-z}}{(1+\tau)^{\sigma}\Phi^2}\right] < 0
\end{aligned}
$$

Firm-level domestic production is:

$$
\begin{aligned}
q_d(\varphi,\tau) &= \left(\frac{\varphi}{\rho}\right)^{\sigma} Q(x_d(\tau))\,P(x_d(\tau))^{\sigma} = \left(\frac{L\varphi^{\sigma}}{\rho^{\sigma}}\right)P(x_d(\tau))^{\sigma-1} = \frac{Lz\varphi^{\sigma}x_d^z}{ab^a\rho\Phi} \\
\frac{\partial q_d(\varphi,\tau)}{\partial\tau} &= \frac{Lz\varphi^{\sigma}}{ab^a\rho}\left[\frac{\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau}-\frac{\partial\Phi}{\partial\tau}x_d^z}{\Phi^2}\right] = \frac{Lz\varphi^{\sigma}x_d^z}{ab^a\rho}\left[\frac{(\sigma-1)\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{a+1}\Phi^2}\right] > 0
\end{aligned}
$$

Firm-level domestic profit is:

$$
\begin{aligned}
\pi_d(\varphi,\tau) &= \left[\frac{Q(x_d)\,P(x_d)^{\sigma}}{\sigma\rho^{\sigma-1}}\right]\varphi^{\sigma-1} - c = \left(\frac{L\varphi^{\sigma-1}}{\sigma\rho^{\sigma-1}}\right)P(x_d)^{\sigma-1} - c = \frac{Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma\Phi} - c \\
\frac{\partial\pi_d(\varphi,\tau)}{\partial\tau} &= \frac{Lz\varphi^{\sigma-1}}{ab^a\sigma}\left[\frac{\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau}-\frac{\partial\Phi}{\partial\tau}x_d^z}{\Phi^2}\right] = \frac{Lz\varphi^{\sigma-1}x_d^z}{ab^a\sigma}\left[\frac{(\sigma-1)\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{a+1}\Phi^2}\right] > 0
\end{aligned}
$$

(b) Aggregate production

Aggregate foreign production is:

$$
\begin{aligned}
Q_f(\tau) &= \int_{x_f}^{\infty} q_f(\varphi) M_f^{in} h_f^{old}(\varphi)\, d\varphi + \int_{y_f}^{\infty} q_f(\varphi) M_f^{out} h_f^{new}(\varphi)\, d\varphi \\
&= \frac{Q(x_d(\tau)) P(x_d(\tau))^{\sigma}}{\rho^{\sigma}(1+\tau)^{\sigma}} \left[\int_{\eta_2(1+\tau)x_d(\tau)}^{\infty} \varphi^{\sigma} g(\varphi)\, d\varphi + \int_{\eta_3(1+\tau)x_d(\tau)}^{\infty} \varphi^{\sigma} g(\varphi)\, d\varphi \right] \\
&= \frac{LP(x_d(\tau))^{\sigma-1}}{\rho^{\sigma}(1+\tau)^{\sigma}} \left[\frac{ab^a}{z-1} (\eta_2(1+\tau)x_d(\tau))^{1-z} + \frac{ab^a}{z-1} (\eta_3(1+\tau)x_d(\tau))^{1-z} \right] \\
&= \frac{Lab^a\left(\eta_2^{1-z}+\eta_3^{1-z}\right)}{\rho^{\sigma}(z-1)(1+\tau)^{a}} P(x_d(\tau))^{\sigma-1} x_d(\tau)^{1-z} = \frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)}{\rho(z-1)} \left[\frac{x_d}{(1+\tau)^a \Phi}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q_f(\tau)}{\partial \tau} &= \frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)}{\rho(z-1)} \left\{ \frac{(1+\tau)^a \Phi \frac{\partial x_d}{\partial \tau} - x_d\left[(1+\tau)^a \frac{\partial \Phi}{\partial \tau} + a(1+\tau)^{a-1}\Phi\right]}{(1+\tau)^{2a}\Phi^2} \right\} \\
&= \frac{Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)x_d}{\rho(z-1)} \left[\frac{(a-1)(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right) - a\Phi}{(1+\tau)^{a+1}\Phi^2} \right] \\
&= \frac{-Lz\left(\eta_2^{1-z}+\eta_3^{1-z}\right)x_d}{\rho(z-1)} \left[\frac{a\left(1+\eta_1^{-z}\right) + (1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{a+1}\Phi^2} \right] < 0
\end{aligned}
$$

Aggregate domestic production is:

$$
\begin{aligned}
Q_d(\tau) &= \int_{x_d}^{\infty} q_d(\varphi) M_d^{in} h_d^{old}(\varphi)\, d\varphi + \int_{y_d}^{\infty} q_d(\varphi) M_d^{out} h_d^{new}(\varphi)\, d\varphi \\
&= \frac{Q(x_d(\tau)) P(x_d(\tau))^{\sigma}}{\rho^{\sigma}} \left[\int_{x_d(\tau)}^{\infty} \varphi^{\sigma} g(\varphi)\, d\varphi + \int_{\eta_1 x_d(\tau)}^{\infty} \varphi^{\sigma} g(\varphi)\, d\varphi \right] \\
&= \frac{LP(x_d(\tau))^{\sigma-1}}{\rho^{\sigma}} \left[\frac{ab^a}{z-1} x_d^{1-z} + \frac{ab^a}{z-1} (\eta_1 x_d(\tau))^{1-z} \right] \\
&= \frac{ab^a L\left(1+\eta_1^{1-z}\right)}{\rho^{\sigma}(z-1)} P(x_d(\tau))^{\sigma-1} x_d^{1-z} = \frac{Lz\left(1+\eta_1^{1-z}\right)}{\rho(z-1)} \left(\frac{x_d}{\Phi}\right)
\end{aligned}
$$

$$
\frac{\partial Q_d(\tau)}{\partial \tau} = \frac{Lz\left(1+\eta_1^{1-z}\right)}{\rho(z-1)} \left(\frac{\Phi \frac{\partial x_d}{\partial \tau} - \frac{\partial \Phi}{\partial \tau} x_d}{\Phi^2} \right) = \frac{Lz\left(1+\eta_1^{1-z}\right)}{\rho(z-1)} \left[\frac{(a-1)\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d}{(1+\tau)^{a+1}\Phi^2} \right] > 0
$$

(c) Average observed productivity

For $i \in \{d, f\}$, the average productivity of firms that are producing in the economy (and hence observed) is:

$$
\begin{aligned}
\overline{\varphi}_i &= \left(\frac{M_i^{in}}{M_i}\right)\int_{x_i}^{\infty}\varphi\left(\frac{g(\varphi)}{1-G(x_i)}\right)d\varphi+\left(\frac{M_i^{out}}{M_i}\right)\int_{y_i}^{\infty}\varphi\left(\frac{g(\varphi)}{1-G(y_i)}\right)d\varphi \\
&= \frac{1}{M_i}\left[\int_{x_i}^{\infty}\varphi g(\varphi)\,d\varphi+\int_{y_i}^{\infty}\varphi g(\varphi)\,d\varphi\right] \\
&= \frac{ab^a}{b^a\left(x_i^{-a}+y_i^{-a}\right)}\left[\int_{x_i}^{\infty}\varphi^{-a}d\varphi+\int_{y_i}^{\infty}\varphi^{-a}d\varphi\right] \\
&= \frac{a}{\left(x_i^{-a}+y_i^{-a}\right)}\left[\lim_{t\to\infty}\int_{x_i}^{t}\varphi^{-a}d\varphi+\lim_{t\to\infty}\int_{y_i}^{t}\varphi^{-a}d\varphi\right] \\
&= \frac{a}{\left(x_i^{-a}+y_i^{-a}\right)}\left\{\lim_{t\to\infty}\left[\frac{\varphi^{1-a}}{1-a}\right]_{x_i}^{t}+\lim_{t\to\infty}\left[\frac{\varphi^{1-a}}{1-a}\right]_{y_i}^{t}\right\} \\
&= \frac{a\left(x_i^{1-a}+y_i^{1-a}\right)}{(a-1)\left(x_i^{-a}+y_i^{-a}\right)}
\end{aligned}
$$

The average observed productivity for foreign firms is:

$$
\begin{aligned}
\overline{\varphi}_f &= \frac{a\left(x_f^{1-a}+y_f^{1-a}\right)}{(a-1)\left(x_f^{-a}+y_f^{-a}\right)}=\frac{a\left(x_f^{1-a}+\left(\frac{\eta_3}{\eta_2}x_f\right)^{1-a}\right)}{(a-1)\left(x_f^{-a}+\left(\frac{\eta_3}{\eta_2}x_f\right)^{-a}\right)}=\frac{ax_f\left[1+\left(\frac{\eta_2}{\eta_3}\right)^{a-1}\right]}{(a-1)\left[1+\left(\frac{\eta_2}{\eta_3}\right)^{a}\right]} \\
\frac{\partial\overline{\varphi}_f}{\partial\tau} &= \frac{a\left[1+\left(\frac{\eta_2}{\eta_3}\right)^{a-1}\right]}{(a-1)\left[1+\left(\frac{\eta_2}{\eta_3}\right)^{a}\right]}\left(\frac{\partial x_f}{\partial\tau}\right)
\end{aligned}
$$

And:

$$
\begin{aligned}
\frac{\partial x_f}{\partial\tau} &= \eta_2\left[(1+\tau)\frac{\partial x_d}{\partial\tau}+x_d\right]=\eta_2\left[x_d-\frac{x_d}{(1+\tau)^a\Phi}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] \\
&= \frac{\eta_2 x_d}{\Phi}\left[\Phi-(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]=\frac{\eta_2 x_d}{\Phi}\left(1+\eta_1^{-z}\right)>0 \quad\Rightarrow\quad \frac{\partial\overline{\varphi}_f}{\partial\tau}>0
\end{aligned}
$$

The average observed productivity for domestic firms is:

$$
\begin{aligned}
\overline{\varphi}_d &= \frac{a\left(x_d^{1-a}+y_d^{1-a}\right)}{(a-1)\left(x_d^{-a}+y_d^{-a}\right)}=\frac{a\left(x_d^{1-a}+(\eta_1 x_d)^{1-a}\right)}{(a-1)\left(x_d^{-a}+(\eta_1 x_d)^{-a}\right)}=\frac{ax_d\left(1+\eta_1^{1-a}\right)}{(a-1)\left(1+\eta_1^{-a}\right)} \\
\frac{\partial\overline{\varphi}_d}{\partial\tau} &= \frac{a\left(1+\eta_1^{1-a}\right)}{(a-1)\left(1+\eta_1^{-a}\right)}\left(\frac{\partial x_d}{\partial\tau}\right)
\end{aligned}
$$

And:

$$
\begin{aligned}
\frac{\partial x_d}{\partial\tau} &= \frac{\psi}{a}\Phi^{\frac{1}{a}-1}\left[-a(1+\tau)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] \\
&= -\psi\Phi^{\frac{1-a}{a}}(1+\tau)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)<0 \quad\Rightarrow\quad \frac{\partial\overline{\varphi}_d}{\partial\tau}<0
\end{aligned}
$$

QED.

Proposition 3. (Existence of a Political Equilibrium)
There exists a political equilibrium in which the host government chooses an optimal takings rate, and firms operate in the resulting market equilibrium.

Proof of Proposition 3.
Suppressing industry subscripts, the government's objective function is:

$$U(\tau) = \tau Q_f(\tau)$$

The first order condition is thus:

$$\begin{aligned}
\frac{\partial U(\tau)}{\partial \tau} &= \tau \frac{\partial Q_f(\tau)}{\partial \tau} + Q_f(\tau) \\
&= -\frac{\tau L z \left(\eta_2^{1-z} + \eta_3^{1-z}\right)\psi}{\rho (z-1)(1+\tau)^{a+1}} \Phi^{\frac{1-2a}{a}} \left[a\left(1+\eta_1^{-z}\right) + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right)\right] + \frac{L z \left(\eta_2^{1-z} + \eta_3^{1-z}\right)\psi}{\rho (z-1)(1+\tau)^{a}} \Phi^{\frac{1-a}{a}} \\
&= \frac{L z \left(\eta_2^{1-z} + \eta_3^{1-z}\right)\psi}{\rho (z-1)(1+\tau)^{a+1}} \Phi^{\frac{1-2a}{a}} \left\{(1+\tau)\Phi - \tau\left[a\left(1+\eta_1^{-z}\right) + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right)\right]\right\} \\
&= \frac{L z \left(\eta_2^{1-z} + \eta_3^{1-z}\right)\psi}{\rho (z-1)(1+\tau)^{a+1}} \Phi^{\frac{1-2a}{a}} \left[(1+\tau-\tau a)\left(1+\eta_1^{-z}\right) + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right)\right]
\end{aligned}$$

Note that:

$$\frac{\partial U}{\partial \tau} \geq 0 \Leftrightarrow 0 \leq (1+\tau-\tau a)\left(1+\eta_1^{-z}\right) + (1+\tau)^{-a}\left(\eta_2^{-z} + \eta_3^{-z}\right) \equiv \Gamma(\tau)$$

Additionally, note that:

$$\Gamma(\tau = 0) = 1 + \eta_1^{-z} + \eta_2^{-z} + \eta_3^{-z} > 0$$

$$\frac{\partial \Gamma(\tau)}{\partial \tau} = -(a-1)\left(1+\eta_1^{-z}\right) - a(1+\tau)^{-a-1}\left(\eta_2^{-z} + \eta_3^{-z}\right) < 0$$

$$\lim_{\tau\to\infty} \Gamma(\tau) = \lim_{\tau\to\infty} \left\{[1-\tau(a-1)]\left(1+\eta_1^{-z}\right) + \frac{\eta_2^{-z} + \eta_3^{-z}}{(1+\tau)^a}\right\} = -\infty < 0$$

So by the intermediate value theorem, there is a unique optimal takings rate. Furthermore:

$$\frac{\partial U(\tau)}{\partial \tau} = 0 \Leftrightarrow \Gamma(\tau) = 0$$

So the political equilibrium for a given industry is characterized by the implicitly defined variable τ_j^*, where τ_j^* solves:

$$\Gamma_j\left(\tau_j^*\right) = \left(1+\tau_j^* - \tau_j^* a\right)\left(1+\eta_{j1}^{-z}\right) + \left(1+\tau_j^*\right)^{-a}\left(\eta_{j2}^{-z}+\eta_{j3}^{-z}\right) = 0$$

This solution in turn determines the equilibrium cutpoints, $\left(x_d^*, y_d^*, x_f^*, y_f^*\right)$. QED.

Lemma. In equilibrium:

$$1 < \tau^*\left(a-1\right)$$

Proof of Lemma.

$$\Gamma\left(\tau^*\right) = 0 \quad \Leftrightarrow \quad 1+\tau^* - \tau^* a = -\frac{\eta_2^{-z}+\eta_3^{-z}}{\left(1+\tau^*\right)^a\left(1+\eta_1^{-z}\right)} < 0 \quad \Leftrightarrow \quad 1 < \tau^*\left(a-1\right)$$

QED.

Proposition 4. (Outcome 1: Takings)
In equilibrium, higher start-up costs for foreign firms lead to a lower takings rate.

Proof of Proposition 4.
Suppressing industry subscripts, recall that the equilibrium taking rate, τ, solves:

$$\Gamma\left(\tau^*\right) = \left(1+\tau^* - \tau^* a\right)\left(1+\eta_1^{-z}\right) + \left(1+\tau^*\right)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right) = 0$$

So by the implicit function theorem:

$$\frac{\partial\tau^*}{\partial y} \quad = \quad \frac{-\Gamma_y}{\Gamma_{\tau^*}} \quad \text{for any exogenous parameter } y$$

Note that:

$$\Gamma_{\tau^*} = -\left[\left(a-1\right)\left(1+\eta_1^{-z}\right) + a\left(1+\tau^*\right)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] < 0$$

And:

$$\Gamma_{\kappa_f} = -\frac{z}{\left(\sigma-1\right)\left(1+\tau\right)^a}\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f} + \frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right) < 0$$

$$\Rightarrow \frac{\partial\tau^*}{\partial\kappa_f} = \frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}} = \frac{-z\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f} + \frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)}{\left(\sigma-1\right)\left(1+\tau\right)^a\left[\left(a-1\right)\left(1+\eta_1^{-z}\right) + a\left(1+\tau^*\right)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]} < 0$$

QED.

Proposition 5. (Outcome 2: Productivity) In equilibrium:

(a) higher start-up costs lead to higher average productivity for foreign firms when mobility is sufficiently high or the per period fixed production cost is sufficiently low;

(b) foreign firms are more productive on average than domestic firms; and

(c) new foreign firms are more productive on average than old foreign firms.

Proof of Proposition 5.

(a) As shown in the Proof for Proposition 2, the average observed productivity for foreign firms is:

$$\overline{\varphi}_f = \frac{a(1+\tau^*)x_d^* G}{(a-1)} \quad \text{where: } G \equiv \frac{\eta_2^{1-a} + \eta_3^{1-a}}{\eta_2^{-a} + \eta_3^{-a}}$$

The total effect of foreign start-up costs on $\overline{\varphi}_f$ is:

$$\frac{d\overline{\varphi}_f}{d\kappa_f} = \frac{\partial\overline{\varphi}_f}{\partial\kappa_f} + \left(\frac{\partial\overline{\varphi}_f}{\partial\tau^*}\right)_{(+)}\left(\frac{\partial\tau^*}{\partial\kappa_f}\right)_{(-)}$$

where:

$$\frac{\partial\overline{\varphi}_f}{\partial\kappa_f} = \frac{a(1+\tau^*)}{(a-1)}\left(x_d^*\frac{\partial G}{\partial\kappa_f} + \frac{\partial x_d^*}{\partial\kappa_f}G\right) = \frac{a(1+\tau^*)x_d^*}{(a-1)}\left[\frac{\partial G}{\partial\kappa_f} + \frac{G}{a\Phi^*}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right)\right]$$

$$\begin{aligned}
\frac{\partial\overline{\varphi}_f}{\partial\tau^*} &= \frac{aG}{(a-1)}\left[(1+\tau^*)\frac{\partial x_d^*}{\partial\tau^*} + x_d^*\right] = \frac{aG}{(a-1)}\left[\frac{(1+\tau^*)x_d^*}{a\Phi^*}\left(\frac{\partial\Phi}{\partial\tau^*}^*\right) + x_d^*\right] \\
&= \frac{aGx_d^*}{(a-1)}\left\{\frac{(1+\tau^*)}{a\Phi^*}\left[-a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] + 1\right\} \\
& \frac{aGx_d^*}{(a-1)\Phi^*}\left[\Phi^* - (1+\tau^*)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{a\left(1+\eta_1^{-z}\right)Gx_d^*}{(a-1)\Phi^*}
\end{aligned}$$

Recall that:

$$\Gamma(\tau^*) = (1+\tau^*-\tau^* a)\left(1+\eta_1^{-z}\right) + (1+\tau^*)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right) = 0$$

And:

$$\frac{\partial\tau^*}{\partial\kappa_f} = \frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}} = \frac{-1}{\Gamma_{\tau^*}}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right)$$

So:

$$\begin{aligned}
\frac{d\overline{\varphi}_f}{d\kappa_f} &= \frac{a(1+\tau^*)x_d^*}{(a-1)}\left[\frac{\partial G}{\partial\kappa_f} + \frac{G}{a\Phi^*}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right)\right] - \frac{a\left(1+\eta_1^{-z}\right)Gx_d^*}{(a-1)\Phi^*\Gamma_{\tau^*}}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right) \\
&= \frac{a(1+\tau^*)x_d^*}{(a-1)}\left[\frac{\partial G}{\partial\kappa_f} + \frac{G}{a\Phi^*}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right) - \frac{\left(1+\eta_1^{-z}\right)G}{(1+\tau^*)\Phi^*\Gamma_{\tau^*}}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right)\right] \\
&= \frac{a(1+\tau^*)x_d^*}{(a-1)}\left\{\frac{\partial G}{\partial\kappa_f} + \frac{G}{a\Phi^*}\left[\frac{(1+\tau^*)\Gamma_{\tau^*} - a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right)\right\}
\end{aligned}$$

Recall that the equilibrium tax rate, τ^*, solves:

$$\Gamma(\tau^*) = 0 \quad \Leftrightarrow \quad (1+\tau^*)^a \left[(a-1)\tau^* - 1\right]\left(1+\eta_1^{-z}\right) = \eta_2^{-z} + \eta_3^{-z}$$

So:

$$\Gamma_{\tau^*} = -\left[(a-1)\left(1+\eta_1^{-z}\right) + a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{-\left[\left(a^2-1\right)\tau^* - a\right]\left(1+\eta_1^{-z}\right)}{(1+\tau^*)}$$

$$(1+\tau^*)\Gamma_{\tau^*} = -\left[\left(a^2-1\right)\tau^* - a\right]\left(1+\eta_1^{-z}\right)$$

$$(1+\tau^*)\Gamma_{\tau^*} - a\left(1+\eta_1^{-z}\right) = -\left(a^2-1\right)\tau^*\left(1+\eta_1^{-z}\right)$$

$$\Phi^* = (a-1)\tau^*\left(1+\eta_1^{-z}\right) \quad \Rightarrow \quad \frac{\partial \Phi^*}{\partial \kappa_f} = (a-1)\tau^*\left(-z\eta_1^{-z-1}\frac{\partial \eta_1}{\partial \kappa_f}\right) = 0$$

So:

$$\frac{d\overline{\varphi}_f}{d\kappa_f} \quad = \quad \frac{d\overline{\varphi}_f}{d\kappa_f} \geq 0 \frac{a(1+\tau^*)x_d^*}{(a-1)}\left(\frac{\partial G}{\partial \kappa_f}\right) \quad \Rightarrow \quad \Leftrightarrow 0 \leq \frac{\partial G}{\partial \kappa_f}$$

Define:

$$\gamma(x) = \eta_2^{x-1}\frac{\partial \eta_2}{\partial \kappa_f} + \eta_3^{x-1}\frac{\partial \eta_3}{\partial \kappa_f} = \frac{1}{\sigma-1}\left(\frac{\mu_f \eta_2^x}{\mu_f \kappa_f + \alpha_f} + \frac{\eta_3^x}{\kappa_f + \alpha_f}\right) > 0$$

Note that:

$$\frac{\partial\left(\eta_2^{1-a}+\eta_3^{1-a}\right)}{\partial \kappa_f} = (1-a)\left[\eta_2^{-a}\left(\frac{\partial \eta_2}{\partial \kappa_f}\right) + \eta_3^{-a}\left(\frac{\partial \eta_3}{\partial \kappa_f}\right)\right] = -(a-1)\gamma(1-a)$$
$$\frac{\partial\left(\eta_2^{-a}+\eta_3^{-a}\right)}{\partial \kappa_f} = -a\left[\eta_2^{-a-1}\left(\frac{\partial \eta_2}{\partial \kappa_f}\right) + \eta_3^{-a-1}\left(\frac{\partial \eta_3}{\partial \kappa_f}\right)\right] = -a\gamma(-a)$$

So:

$$\frac{\partial G}{\partial \kappa_f} \quad = \quad \frac{a\gamma(-a)\left(\eta_2^{1-a}+\eta_3^{1-a}\right) - (a-1)\left(\eta_2^{-a}+\eta_3^{-a}\right)\gamma(1-a)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)^2}$$

Note that:

$$\frac{\partial G}{\partial \kappa_f} \geq 0 \quad \Leftrightarrow \quad 0 \leq a\gamma(-a)\left(\eta_2^{1-a}+\eta_3^{1-a}\right) - (a-1)\left(\eta_2^{-a}+\eta_3^{-a}\right)\gamma(1-a) \equiv J$$

Foreign mobility
Note that J is continuous in μ_f and when $\mu_f = 1$, $\eta_2 = \eta_3 \equiv \widehat{\eta}$. So:

$$\gamma\left(x|\mu_f=1\right)=\frac{2\widehat{\eta}^{x}}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}$$

$$\begin{aligned} J\left(\mu_f=1\right) &= 2a\left[\frac{2\widehat{\eta}^{-a}}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\right]\widehat{\eta}^{1-a}-2\left(a-1\right)\left[\frac{2\widehat{\eta}^{1-a}}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\right]\widehat{\eta}^{-a} \\ &= \frac{4}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}\left\{a\widehat{\eta}^{1-2a}-\left(a-1\right)\widehat{\eta}^{1-2a}\right\}=\frac{4\widehat{\eta}^{1-2a}}{\left(\sigma-1\right)\left(\kappa_f+\alpha_f\right)}>0 \end{aligned}$$

So $J>0$ for sufficiently large values of μ_f.

Per period production costs

Recall that $\alpha_f=c-\delta\Psi_f>0$, which is increasing in c. Note that J is continuous in α_f and:

$$\begin{aligned} J\left(\alpha_f=0\right) &= \frac{a}{\left(\sigma-1\right)\kappa_f}\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{1-a}+\eta_3^{1-a}\right)-\frac{\left(a-1\right)}{\left(\sigma-1\right)\kappa_f}\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{1-a}+\eta_3^{1-a}\right) \\ &= \frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{1-a}+\eta_3^{1-a}\right)}{\left(\sigma-1\right)\kappa_f}>0 \end{aligned}$$

So $J>0$ for sufficiently small values of c.

(b) As shown in the Proof for Proposition 2, the average observed productivity for domestic firms is:

$$\overline{\varphi}_d = \frac{a\left(x_d^{1-a}+y_d^{1-a}\right)}{\left(a-1\right)\left(x_d^{-a}+y_d^{-a}\right)}=\frac{ax_d\left(1+\eta_1^{1-a}\right)}{\left(a-1\right)\left(1+\eta_1^{-a}\right)}$$

So:

$$\begin{aligned} \overline{\varphi}_d<\overline{\varphi}_f &\Leftrightarrow \frac{ax_d\left(1+\eta_1^{1-a}\right)}{\left(a-1\right)\left(1+\eta_1^{-a}\right)}<\frac{a\left(1+\tau\right)x_d\left(\eta_2^{1-a}+\eta_3^{1-a}\right)}{\left(a-1\right)\left(\eta_2^{-a}+\eta_3^{-a}\right)} \\ &\Leftrightarrow \left[1+\left(\frac{1}{\eta_1}\right)^{a-1}\right]\left[\left(\frac{1}{\eta_2}\right)^{a}+\left(\frac{1}{\eta_3}\right)^{a}\right]<\left(1+\tau\right)\left[1+\left(\frac{1}{\eta_1}\right)^{a}\right]\left[\left(\frac{1}{\eta_2}\right)^{a-1}+\left(\frac{1}{\eta_3}\right)^{a-1}\right] \\ &\Leftrightarrow \left(\eta_1^a+\eta_1\right)\left(\eta_2^a+\eta_3^a\right)<\left(1+\tau\right)\left(\eta_1^a+1\right)\left(\eta_2\eta_3^a+\eta_2^a\eta_3\right) \\ &\Leftrightarrow 0<\left[\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_2-\left(\eta_1^a+\eta_1\right)\right]\eta_3^a+\left[\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_3-\left(\eta_1^a+\eta_1\right)\right]\eta_2^a \end{aligned}$$

Note that:

$$\mu_d\kappa_d+\alpha_d<\mu_f\kappa_f+\alpha_f\Leftrightarrow 1<\eta_2\Leftrightarrow\left(1+\tau\right)\left(\eta_1^a+1\right)<\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_2\Leftrightarrow 0<\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_2-\left(1+\tau\right)\left(\eta_1^a+1\right)\equiv y$$

$$\kappa_d+\alpha_d<\kappa_f+\alpha_f\Leftrightarrow\eta_1<\eta_3\Leftrightarrow\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_1<\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_3\Leftrightarrow 0<\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_3-\left(1+\tau\right)\left(\eta_1^a+1\right)\eta_1\equiv w$$

So:

$$\begin{aligned}
\overline{\varphi}_d < \overline{\varphi}_f \quad \Leftrightarrow \quad & 0 < \left[y + (1+\tau)\left(\eta_1^a + 1\right) - \left(\eta_1^a + \eta_1\right)\right]\eta_3^a + \left[w + (1+\tau)\left(\eta_1^a + 1\right)\eta_1 - \left(\eta_1^a + \eta_1\right)\right]\eta_2^a \\
& 0 < y\eta_3^a + w\eta_2^a + \left[(1+\tau)\left(\eta_1^a + 1\right) - \left(\eta_1^a + \eta_1\right)\right]\eta_3^a + \left[(1+\tau)\left(\eta_1^a + 1\right)\eta_1 - \left(\eta_1^a + \eta_1\right)\right]\eta_2^a \\
& 0 < y\eta_3^a + w\eta_2^a + \left[\left(\tau\eta_1^{a-1} - 1\right)_{(+)}\eta_1 + 1 + \tau\right]\eta_3^a + \left\{\left[(1+\tau)\eta_1 - 1\right]_{(+)}\eta_1^a + \tau\eta_1\right\}\eta_2^a
\end{aligned}$$

This holds for any value of τ because $1 < \eta_1$ and $1 < a$.

(c) Note that:

$$\begin{aligned}
\overline{\varphi}_f^{old} &= \int_{x_f}^{\infty} \varphi\left(\frac{g(\varphi)}{1 - G(x_f)}\right) d\varphi = ab^a \left(\frac{x_f}{b}\right)^a \lim_{t\to\infty} \int_{x_f}^{t} \varphi^{-a} d\varphi \\
&= ax_f^a \lim_{t\to\infty}\left[\frac{\varphi^{1-a}}{1-a}\right]_{x_f}^{t} = \frac{-ax_f^a}{a-1}\lim_{t\to\infty}\left[\frac{1}{\varphi^{a-1}}\right]_{x_f}^{t} = \frac{ax_f}{a-1} \\
\overline{\varphi}_f^{new} &= \int_{y_f}^{\infty} \varphi\left(\frac{g(\varphi)}{1 - G(y_f)}\right) d\varphi = ab^a \left(\frac{y_f}{b}\right)^a \lim_{t\to\infty} \int_{y_f}^{t} \varphi^{-a} d\varphi \\
&= ay_f^a \lim_{t\to\infty}\left[\frac{\varphi^{1-a}}{1-a}\right]_{y_f}^{t} = \frac{-ay_f^a}{a-1}\lim_{t\to\infty}\left[\frac{1}{\varphi^{a-1}}\right]_{y_f}^{t} = \frac{ay_f}{a-1}
\end{aligned}$$

So the following always holds.

$$\overline{\varphi}_f^{old} < \overline{\varphi}_f^{new} \quad \Leftrightarrow \quad x_f < y_f \quad \Leftrightarrow \quad \eta_2 < \eta_3$$

QED.

Proposition 6. (Outcome 3: Revenue) In equilibrium:

(a) higher start-up costs are associated with higher revenue for foreign firms;
(b) foreign firms have higher average revenue than domestic firms; and
(c) new foreign firms have higher average revenue than old foreign firms.

Proof of Proposition 6

Foreign firms

In equilibrium, the revenue of a foreign firm is:

$$r_f^*(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} LP(x_d)^{\sigma-1} = \frac{zLx_d^z\varphi^{\sigma-1}}{ab^a(1+\tau)^{\sigma-1}\Phi}$$

And the profit of a foreign firm is: $\pi_f^*(\varphi) = \frac{r_f^*(\varphi)}{\sigma} - c$

The average revenue of an old foreign firm is:

$$\begin{aligned}
\overline{r}_f^{old} &= \frac{zLx_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi}\int_{x_f}^{\infty}\varphi^{\sigma-1}h_f^{old}d\varphi \quad \text{where: } h_f^{old}=\frac{g(\varphi)}{1-G(x_f)} \\
&= \frac{zLx_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi[1-G(x_f)]}\left(\frac{ab^a}{z}x_f^{-z}\right) \\
&= \frac{Lx_d^z x_f^{\sigma-1}}{(1+\tau)^{\sigma-1}\Phi b^a}=\frac{L\eta_2^{\sigma-1}x_d^a}{b^a\Phi}
\end{aligned}$$

And the average profit of an old foreign firm is: $\overline{\pi}_f^{old}=\frac{\overline{r}_f^{old}}{\sigma}-c$.
The average revenue of a new foreign firm is:

$$\begin{aligned}
\overline{r}_f^{new} &= \frac{zLx_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi}\int_{y_f}^{\infty}\varphi^{\sigma-1}h_f^{new}d\varphi \quad \text{where: } h_f^{new}=\frac{g(\varphi)}{1-G(y_f)} \\
&= \frac{zLx_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi[1-G(y_f)]}\left(\frac{ab^a}{z}y_f^{-z}\right) \\
&= \frac{Lx_d^z y_f^{\sigma-1}}{b^a(1+\tau)^{\sigma-1}\Phi}=\frac{L\eta_3^{\sigma-1}x_d^a}{b^a\Phi}
\end{aligned}$$

And the average profit of a new foreign firm is: $\overline{\pi}_f^{new}=\frac{\overline{r}_f^{new}}{\sigma}-c$.
So the average revenue of a foreign firm is:

$$\begin{aligned}
\overline{r}_f &= \frac{M_f^{in}\overline{r}_f^{old}}{M_f}+\frac{M_f^{out}\overline{r}_f^{new}}{M_f}=\frac{1}{M_f}\left\{[1-G(x_f)]\left(\frac{L\eta_2^{\sigma-1}x_d^a}{b^a\Phi}\right)+[1-G(y_f)]\left(\frac{L\eta_3^{\sigma-1}x_d^a}{b^a\Phi}\right)\right\} \\
&= \frac{(1+\tau)^a x_d^a}{b^a\left(\eta_2^{-a}+\eta_3^{-a}\right)}\left(\frac{Lx_d^a}{b^a\Phi}\right)\left(b^a x_f^{-a}\eta_2^{\sigma-1}+b^a y_f^{-a}\eta_3^{\sigma-1}\right)=\frac{L\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d^a}{b^a\left(\eta_2^{-a}+\eta_3^{-a}\right)\Phi}
\end{aligned}$$

And the average profit of a foreign firm is: $\overline{\pi}_f=\frac{\overline{r}_f}{\sigma}-c$.

Domestic firms

In equilibrium, the revenue of a domestic firm is:

$$r_d^*(\varphi)=\left(\frac{\varphi}{\rho}\right)^{\sigma-1}QP^\sigma=\left(\frac{\varphi}{\rho}\right)^{\sigma-1}LP(x_d^*)^{\sigma-1}=\frac{zLx_d^z\varphi^{\sigma-1}}{ab^a\Phi}$$

And the profit of a domestic firm is: $\pi_d^*(\varphi)=\frac{r_d^*(\varphi)}{\sigma}-c$
The average revenue of an old domestic firm is:

$$\begin{aligned}
\overline{r}_d^{old} &= \frac{zLx_d^z}{ab^a\Phi}\int_{x_d}^{\infty}\varphi^{\sigma-1}h_d^{old}d\varphi \quad \text{where: } h_d^{old}=\frac{g(\varphi)}{1-G(x_d)} \\
&= \frac{zLx_d^z}{ab^a\Phi[1-G(x_d)]}\left(\frac{ab^a}{z}x_d^{-z}\right)=\frac{Lx_d^a}{b^a\Phi}
\end{aligned}$$

And the average profit of an old domestic firm is: $\overline{\pi}_d^{old}=\frac{\overline{r}_d^{old}}{\sigma}-c$.

The average revenue of a new domestic firm is:

$$\begin{aligned}
\overline{r}_d^{new} &= \frac{zLx_d^z}{ab^a\Phi}\int_{y_d}^{\infty}\varphi^{\sigma-1}h_d^{new}d\varphi \quad \text{where: } h_d^{new}=\frac{g(\varphi)}{1-G(y_d)} \\
&= \frac{zLx_d^z}{ab^a\Phi\left[1-G(y_d)\right]}\left(\frac{ab^a}{z}y_d^{-z}\right)=\frac{Lx_d^z y_d^{\sigma-1}}{b^a\Phi}=\frac{L\eta_1^{\sigma-1}x_d^a}{b^a\Phi}
\end{aligned}$$

And the average profit of a new domestic firm is: $\overline{\pi}_d^{new}=\frac{\overline{r}_d^{new}}{\sigma}-c$.
So the average revenue of a domestic firm is:

$$\begin{aligned}
\overline{r}_d &= \frac{M_d^{in}\overline{r}_d^{old}}{M_d}+\frac{M_d^{out}\overline{r}_d^{new}}{M_d}=\frac{1}{M_d}\left\{\left[1-G(x_d)\right]\left(\frac{Lx_d^a}{b^a\Phi}\right)+\left[1-G(y_d)\right]\left(\frac{L\eta_1^{\sigma-1}x_d^a}{b^a\Phi}\right)\right\} \\
&= \frac{L\left(1+\eta_1^{-z}\right)x_d^a}{b^a\left(1+\eta_1^{-a}\right)\Phi}
\end{aligned}$$

And the average profit of a domestic firm is: $\overline{\pi}_d=\frac{\overline{r}_d}{\sigma}-c$.

(a) The total effect of start-up costs on foreign revenue will be:

$$\frac{dr_f(\varphi)}{d\kappa_f}=\frac{\partial r_f(\varphi)}{\partial\kappa_f}_{(+)}+\frac{\partial r_f(\varphi)}{\partial\tau^*}_{(-)}\left(\frac{\partial\tau^*}{\partial\kappa_f}\right)_{(-)}$$

where:

$$\begin{aligned}
\frac{\partial r_f(\varphi)}{\partial\tau^*} &= \frac{zL\varphi^{\sigma-1}}{ab^a}\left[\frac{(1+\tau)^{\sigma-1}\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\tau}-x_d^z\left[(1+\tau)^{\sigma-1}\frac{\partial\Phi}{\partial\tau}+(\sigma-1)(1+\tau)^{\sigma-2}\Phi\right]}{(1+\tau)^{2(\sigma-1)}\Phi^2}\right] \\
&= \frac{-zL\varphi^{\sigma-1}x_d^z}{ab^a}\left[\frac{z(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)-a(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)+(\sigma-1)\Phi}{(1+\tau)^{\sigma}\Phi^2}\right] \\
&= \frac{-z(\sigma-1)L\varphi^{\sigma-1}x_d^z}{ab^a}\left[\frac{\Phi-(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau)^{\sigma}\Phi^2}\right] \\
&= \frac{-z(\sigma-1)L\left(1+\eta_1^{-z}\right)x_d^z\varphi^{\sigma-1}}{ab^a(1+\tau)^{\sigma}\Phi^2}<0
\end{aligned}$$

And:

$$\frac{\partial\tau^*}{\partial\kappa_f}=\frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}}=\frac{-z\left(\frac{\eta_2^{-z}\mu_f}{\mu_f\kappa_f+\alpha_f}+\frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right)}{(\sigma-1)(1+\tau)^a\left[(a-1)\left(1+\eta_1^{-z}\right)+a(1+\tau^*)^{-a-1}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right]}<0$$

And:

$$\frac{\partial r_f(\varphi)}{\partial\kappa_f}=\frac{zL\varphi^{\sigma-1}}{ab^a(1+\tau)^{\sigma-1}}\left[\frac{\Phi z x_d^{z-1}\frac{\partial x_d}{\partial\kappa_f}-x_d^z\frac{\partial\Phi}{\partial\kappa_f}}{\Phi^2}\right]=\frac{-zL\varphi^{\sigma-1}x_d^z}{ab^a(1+\tau)^{\sigma-1}\Phi^2}\left(\frac{\sigma-1}{a}\right)\frac{\partial\Phi}{\partial\kappa_f}>0$$

So: $\frac{dr_f(\varphi)}{d\kappa_f}>0$ and $\frac{d\pi_f(\varphi)}{d\kappa_f}>0$.

(b) Recall that $a - z = \sigma - 1 > 0$

$$\begin{aligned}
\overline{r}_d < \overline{r}_f \quad &\Leftrightarrow \quad \frac{L\left(1+\eta_1^{-z}\right)x_d^a}{b^a\left(1+\eta_1^{-a}\right)\Phi} < \frac{L\left(\eta_2^{-z}+\eta_3^{-z}\right)x_d^a}{b^a\left(\eta_2^{-a}+\eta_3^{-a}\right)\Phi} \quad \Leftrightarrow \quad \frac{1+\eta_1^{-z}}{1+\eta_1^{-a}} < \frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}} \\
&\Leftrightarrow \quad \frac{\eta_1^a+\eta_1^{\sigma-1}}{\eta_1^a+1} < \frac{\eta_2^{\sigma-1}\eta_3^a+\eta_2^a\eta_3^{\sigma-1}}{\eta_2^a+\eta_3^a} \\
&\Leftrightarrow \quad \left(\eta_1^a+\eta_1^{\sigma-1}\right)\left(\eta_2^a+\eta_3^a\right) < \left(1+\eta_1^a\right)\left(\eta_2^{\sigma-1}\eta_3^a+\eta_2^a\eta_3^{\sigma-1}\right) \\
&\Leftrightarrow \quad 0 < \left(1+\eta_1^a\right)\left(\eta_2^{\sigma-1}\eta_3^a+\eta_2^a\eta_3^{\sigma-1}\right) - \left(\eta_1^a+\eta_1^{\sigma-1}\right)\left(\eta_2^a+\eta_3^a\right) \\
&\qquad\quad = \left[\left(1+\eta_1^a\right)\eta_3^{\sigma-1} - \left(\eta_1^a+\eta_1^{\sigma-1}\right)\right]\eta_2^a + \left[\left(1+\eta_1^a\right)\eta_2^{\sigma-1} - \left(\eta_1^a+\eta_1^{\sigma-1}\right)\right]\eta_3^a
\end{aligned}$$

Note that:

$$\eta_1 < \eta_3 \Leftrightarrow \left(1+\eta_1^a\right)\eta_1^{\sigma-1} < \left(1+\eta_1^a\right)\eta_3^{\sigma-1} \Leftrightarrow 0 < \left(1+\eta_1^a\right)\eta_3^{\sigma-1} - \left(1+\eta_1^a\right)\eta_1^{\sigma-1} = y$$

$$\eta_1 < \eta_2 \Leftrightarrow \left(1+\eta_1^a\right)\eta_1^{\sigma-1} < \left(1+\eta_1^a\right)\eta_2^{\sigma-1} \Leftrightarrow 0 < \left(1+\eta_1^a\right)\eta_2^{\sigma-1} - \left(1+\eta_1^a\right)\eta_1^{\sigma-1} = w$$

So:

$$\begin{aligned}
\overline{r}_d < \overline{r}_f \quad \Leftrightarrow \quad 0 &< \left[y + \left(1+\eta_1^a\right)\eta_1^{\sigma-1} - \left(\eta_1^a+\eta_1^{\sigma-1}\right)\right]\eta_2^a + \left[w + \left(1+\eta_1^a\right)\eta_1^{\sigma-1} - \left(\eta_1^a+\eta_1^{\sigma-1}\right)\right]\eta_3^a \\
&= y\eta_2^a + w\eta_3^a + \eta_1^a\left(\eta_1^{\sigma-1}-1\right)\eta_2^a + \eta_1^a\left(\eta_1^{\sigma-1}-1\right)\eta_3^a
\end{aligned}$$

This always holds, which in turn implies that $\overline{\pi}_d < \overline{\pi}_f$.

(c) Note that the following always holds:

$$\overline{r}_f^{old} < \overline{r}_f^{new} \quad \Leftrightarrow \quad \frac{L\eta_2^{\sigma-1}x_d^a}{b^a\Phi} < \frac{L\eta_3^{\sigma-1}x_d^a}{b^a\Phi} \quad \Leftrightarrow \quad \eta_2 < \eta_3$$

So $\overline{\pi}_f^{old} < \overline{\pi}_f^{new}$ also. QED.